Approximation Results for Sums of Independent Random Variables

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Abstract

In this article, we consider Poisson and Poisson convoluted geometric approximation to the sums of n independent random variables under moment conditions. We use Stein's method to derive the approximation results in total variation distance. The error bounds obtained are either comparable to or improvement over the existing bounds available in the literature. Also, we give an application to the waiting time distribution of 2-runs.

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1 Introduction

Let $\xi_1, \xi_2, \ldots, \xi_n$ be n independent random variables concentrated on $\mathbb{Z}_+ = \{0, 1, 2, \ldots\}$ and

$$W_n := \sum_{i=1}^{n} \xi_i, \tag{1}$$

their convolution of n independent random variables. The distribution of W_n has received special attention in the literature due to its applicability in many settings such as rare events, the waiting time distributions, wireless communications, counts in nuclear decay, and business situations, among many others. For large values of n, it is in practice hard to obtain the exact distribution of W_n in general, in fact, it becomes intractable if the underlying distribution is complicated such as hyper-geometric and logarithmic series distribution, among many others. It is therefore of interest to approximate the distribution of such W_n with some well-known and easy to use distributions. Approximations to W_n have been studied by several authors such as, saddle point approximation (Lugannani and Rice (1980) and Murakami (2015)), compound Poisson approximation (Barbour *et al.* (1992a), Serfozo (1986), and Roos (2003)), Poisson approximation (Barbour *et al.* (1992b)), the centred Poisson approximation (Čekanavičius and Vaitkus (2001)), compound negative binomial approximation (Vellaisamy and Upadhye (2009)), and negative

binomial approximation (Vellaisamy *et al.* (2013) and Kumar and Upadhye (2017)).
In this article, we consider Poisson and Poisson convoluted geometric approximation to W_n. Let X and Y follow Poisson and geometric distribution with parameter λ and $p = 1-q$ with probability mass function (PMF)

$$P(X=k) = \frac{e^{-\lambda}\lambda^k}{k!} \quad \text{and} \quad P(Y=k) = q^k p, \quad k = 0,1,2,\dots, \tag{2}$$

respectively. Also, assume X and Y are independent. We use Stein's method to obtain bounds for the approximation of the law of W_n with that of X and $X+Y$. Stein's method (Stein (1986)) requires identification of a Stein operator and there are several approaches to obtain Stein operators (see Reinert (2005)) such as density approach (Stein (1986), Stein *et al.* (2004), Ley and Swan (2013a, 2013b)), generator approach (Barbour (1990) and Götze (1991)), orthogonal polynomial approach (Diaconis and Zabell (1991)), and probability generating function (PGF) approach (Upadhye *et al.* (2014)). We use the PGF approach to obtain Stein operators.
This article is organized as follows. In Section 2, we introduce some notations to simplify the presentation of the article. Also, we discuss some known results of Stein's method. In Section 3, Stein operators for W_n and $X+Y$ are obtained as a perturbation of the Poisson operator. In Section 4, the error bounds for X and $X+Y$ approximation to W_n are derived in total variation distance. In Section 5, we demonstrate the relevance of our results through an application to the waiting time distribution of 2-runs. In Section 6, we point out some relevant remarks.

2 Notations and Preliminaries

Recall that $W_n = \sum_{i=1}^n \xi_i$, where $\xi_1, \xi_2, \dots, \xi_n$ are n independent random variables concentrated on $\mathbb{Z}_+$. Throughout, we assume that ψ_{ξ_i}, the PGF of ξ_i, satisfies

$$\frac{\psi'_{\xi_i}(w)}{\psi_{\xi_i}(w)} = \sum_{j=0}^{\infty} g_{i,j+1} w^j =: \phi_{\xi_i}(w), \tag{3}$$

at all $w \in \mathbb{Z}_+$. Note that this assumption is satisfied for the series (3) converges absolutely. Also, one can show that the hyper-geometric and logarithmic series distribution do not satisfy (3). See Yakshyavichus (1998), and Kumar and Upadhye (2017) for more details. Note that

1. If $\xi_i \sim Po(\lambda_i) \implies g_{i,j+1} = \begin{cases} \lambda_i, & \text{for } j=0, \\ 0, & \text{for } j \geq 1. \end{cases}$
2. If $\xi_i \sim Ge(p_i) \implies g_{i,j+1} = q_i^{j+1}$.
3. If $\xi_i \sim Bi(n,p_i) \implies g_{i,j+1} = n(-1)^j \left(p_i/(1-p_i)\right)^{j+1}$.

Next, let μ and σ^2 be the mean and variance of W_n, respectively. Also, let μ_2 and μ_3 denote the second and third factorial cumulant moments of W_n, respectively. Then, it can be easily verified that

$$\mu = \sum_{i=1}^{n} \phi_{\xi_i}(1) = \sum_{i=1}^{n}\sum_{j=0}^{\infty} g_{i,j+1},\ \sigma^2 = \sum_{i=1}^{n}\left[\phi_{\xi_i}(1) + \phi'_{\xi_i}(1)\right] = \sum_{i=1}^{n}\sum_{j=0}^{\infty}(j+1)g_{i,j+1}, \tag{4}$$

$$\mu_2 = \sum_{i=1}^{n} \phi'_{\xi_i}(1) = \sum_{i=1}^{n}\sum_{j=0}^{\infty} j g_{i,j+1},\ \text{and}\ \mu_3 = \sum_{i=1}^{n} \phi''_{\xi_i}(1) = \sum_{i=1}^{n}\sum_{j=0}^{\infty} j(j-1)g_{i,j+1}.$$

For more details, see Vellaisamy *et al.* (2013), and Kumar and Upadhye (2017).
Next, let $H := \{f | f : \mathbb{Z}_+ \to \mathbb{R} \text{ is bounded}\}$ and

$$H_{\bar{X}} := \{h \in H | h(0) = 0,\ \text{and}\ h(j) = 0 \text{ for } j \notin Supp(\bar{X})\} \tag{5}$$

for a random variable $\bar{X}$ and $Supp(\bar{X})$ denotes the support of random variable $\bar{X}$.
Now, we discuss Stein's method which can be carried out in the following three steps.
We first identify a suitable operator $\mathcal{A}_{\bar{X}}$ for a random variable $\bar{X}$ (known as Stein operator) such that

$$\mathbb{E}(\mathcal{A}_{\bar{X}} h(\bar{X})) = 0, \quad \text{for } h \in H.$$

In the second step, we find a solution to the Stein equation

$$\mathcal{A}_{\bar{X}} h(j) = f(j) - \mathbb{E}f(\bar{X}),\ j \in \mathbb{Z}_+ \text{ and } f \in H_{\bar{X}} \tag{6}$$

and obtain the bound for $\|\Delta h\|$, where $\|\Delta h\| = \sup_{j\in\mathbb{Z}_+} |\Delta h(j)|$ and $\Delta h(j) = h(j+1) - h(j)$ denotes the first forward difference operator.
Finally, substitute a random variable $\bar{Y}$ for j in (6) and taking expectation and supremum, the expression leads to

$$d_{TV}(\bar{X}, \bar{Y}) := \sup_{f\in\mathcal{H}} \left|\mathbb{E}f(\bar{X}) - \mathbb{E}f(\bar{Y})\right| = \sup_{f\in\mathcal{H}} \left|\mathbb{E}[\mathcal{A}_{\bar{X}} h(\bar{Y})]\right|, \tag{7}$$

where $\mathcal{H} = \{\mathbf{1}_A \mid A \subseteq \mathbb{Z}_+\}$ and $\mathbf{1}_A$ is the indicator function of A. Equivalently, (7) can be represented as

$$d_{TV}(\bar{X}, \bar{Y}) = \frac{1}{2}\sum_{j=0}^{\infty} |P(\bar{X} = j) - P(\bar{Y} = j)|.$$

For more details, we refer the reader to Barbour *et. al.* (1992b), Chen *et. al.* (2011), Goldstein and Reinert (2005), and Ross (2011). For recent developments, see Barbour and Chen (2014), Ley *et. al.* (2014), Upadhye *et. al.* (2014), and references therein.
Next, it is known that a Stein operator for $X \sim Po(\lambda)$, the Poisson random variable with parameter λ, is given by

$$\mathcal{A}_X h(j) = \lambda h(j+1) - jh(j), \quad \text{for } j \in \mathbb{Z}_+ \text{ and } h \in H. \tag{8}$$

Also, from Section 5 of Barbour and Eagleson (1983), the bound for the solution to the stein equation (say h_f) is given by

$$\|\Delta h_f\| \leq \frac{1}{\max(1,\lambda)}, \quad \text{for } f \in \mathcal{H},\ h \in H. \tag{9}$$

In terms of $\|f\|$, we have the following bound

$$\|\Delta h_f\| \leq \frac{2\|f\|}{\max(1,\lambda)}, \quad \text{for } f \in \mathcal{H},\ h \in H. \tag{10}$$

See Section 3 of Upadhye *et al.* (2014) for more details. Note that the condition $h(0) = 0$ in (5) is used while obtaining the bound (9), see Barbour and Eagleson (1983) for more details.
Next, suppose we have three random variables X_1, X_2, and X_3 defined on some common probability space. Define $\mathcal{U} = \mathcal{A}_{X_2} - \mathcal{A}_{X_1}$ then the upper bound for $d_{TV}(X_2, X_3)$ can be obtained by the following lemma which is given by Upadhye *et al.* (2014).

Lemma 2.1. *[Lemma 3.1, Upadhye et al. (2014)] Let X_1 be a random variable with support $\mathcal{S}$, Stein operator $\mathcal{A}_{X_1}$, and h_0 be the solution to Stein equation (6) satisfying*

$$\|\Delta h_0\| \leq w_1 \|f\| \min(1, \alpha^{-1}),$$

where $w_1,\ \alpha > 0$. Also, let X_2 be a random variable whose Stein operator can be written as $\mathcal{A}_{X_2} = \mathcal{A}_{X_1} + U_1$ and X_3 be a random variable such that, for $h \in H_{X_1} \cap H_{X_2}$,

$$\|U_1 h\| \leq w_2 \|\Delta h\| \quad \text{and} \quad |\mathbb{E}\mathcal{A}_{X_2} h(X_3)| \leq \varepsilon \|\Delta h\|,$$

where $w_1 w_2 < \alpha$. Then

$$d_{TV}(X_2, X_3) \leq \frac{\alpha}{2(\alpha - w_1 w_2)} \left(\varepsilon w_1 \min(1, \alpha^{-1}) + 2P(X_2 \in \mathcal{S}^c) + 2P(X_3 \in \mathcal{S}^c) \right),$$

where $\mathcal{S}^c$ denote the complement of set $\mathcal{S}$.

Finally, from Corollary 1.6 of Mattner and Roos (2007), we have

$$d_{TV}(W_n, W_n + 1) \leq \sqrt{\frac{2}{\pi}} \left(\frac{1}{4} + \sum_{i=1}^{n} \left(1 - d_{TV}(\xi_i, \xi_i + 1)\right) \right)^{-1/2}. \tag{11}$$

For more details about these results, we refer the reader to Barbour *et al.* (2007), Upadhye *et al.* (2014), Vellaisamy *et al.* (2013), Kumar and Upadhye (2017), and references therein.

3 Stein Operator for the Convolution of Random Variables

In this section, we derive Stein operators for W_n and $X + Y$ as a perturbation of Poisson operator which are used to obtain the main results in Section 4.

Proposition 3.1. *Let $\xi_1, \xi_2, \ldots, \xi_n$ be independent random variables satisfying (3) and $W_n =$*

$\sum_{i=1}^{n} \xi_i$. *Then, a Stein operator for* W_n *is*

$$\mathcal{A}_{W_n} h(j) = \mu h(j+1) - j h(j) + \sum_{i=1}^{n} \sum_{k=0}^{\infty} \sum_{l=1}^{k} g_{i,k+1} \Delta h(j+l),$$

where μ *is defined in* (4).

Proof. It can be easily verified that the PGF of W_n, denoted by ψ_{W_n}, is

$$\psi_{W_n}(w) = \prod_{i=1}^{n} \psi_{\xi_i}(w)$$

as $\xi_1, \xi_2, \ldots, \xi_n$ are independent random variables. Differentiating with respect to w, we have

$$\begin{aligned}\psi'_{W_n}(w) &= \psi_{W_n}(w) \sum_{i=1}^{n} \phi_{\xi_i}(w) \\ &= \sum_{i=1}^{n} \psi_{W_n}(w) \sum_{j=0}^{\infty} g_{i,j+1} w^j,\end{aligned}$$

where $\phi_{\xi_i}(\cdot)$ is defined in (3). Using definition of the PGF, the above expression can be expressed as

$$\sum_{j=0}^{\infty} (j+1)\gamma_{j+1} w^j = \sum_{i=1}^{n} \sum_{k=0}^{\infty} \gamma_k w^k \sum_{j=0}^{\infty} g_{i,j+1} w^j = \sum_{j=0}^{\infty} \left(\sum_{i=1}^{n} \sum_{k=0}^{j} \gamma_k g_{i,j-k+1} \right) w^j,$$

where $\gamma_j = P(W_n = j)$. Comparing the coefficients of w^j, we get

$$\sum_{i=1}^{n} \sum_{k=0}^{j} \gamma_k g_{i,j-k+1} - (j+1)\gamma_{j+1} = 0.$$

Let $h \in H_{W_n}$ as defined in (5), then

$$\sum_{j=0}^{\infty} h(j+1) \left[\sum_{i=1}^{n} \sum_{k=0}^{j} \gamma_k g_{i,j-k+1} - (j+1)\gamma_{j+1} \right] = 0.$$

Therefore,

$$\sum_{j=0}^{\infty} \left[\sum_{i=1}^{n} \sum_{k=0}^{\infty} g_{i,k+1} h(j+k+1) - j h(j) \right] \gamma_j = 0.$$

Hence, a Stein operator for W_n is given by

$$\mathcal{A}_{W_n} h(j) = \sum_{i=1}^{n} \sum_{k=0}^{\infty} g_{i,k+1} h(j+k+1) - j h(j). \tag{12}$$

It is well known that

$$h(j+k+1) = \sum_{l=1}^{k} \Delta h(j+l) + h(j+1). \tag{13}$$

Using (13) in (12), the proof follows. □

Proposition 3.2. *Let $X \sim Po(\lambda)$ and $Y \sim Ge(p)$ as defined in* (2)*. Also, assume X and Y are independent random variables. Then a Stein operator for $X+Y$ is given by*

$$\bar{\mathcal{A}}_{X+Y}h(j) = \Big(\lambda + \frac{q}{p}\Big)h(j+1) - jh(j) + \sum_{k=0}^{\infty}\sum_{l=1}^{k} q^{k+1}\Delta h(j+l).$$

Proof. It is known that the PGF of X and Y are

$$\psi_X(w) = e^{-\lambda(1-w)} \quad \text{and} \quad \psi_Y(w) = \frac{p}{1-qw},$$

respectively. Then, the PGF of $Z = X+Y$ is given by

$$\psi_Z(w) = \psi_X(w).\psi_Y(w).$$

Differentiating with respect to w, we get

$$\psi_Z'(w) = \Big(\lambda + \frac{q}{1-qw}\Big)\psi_Z(w) = \Big(\lambda + q\sum_{j=0}^{\infty} q^j w^j\Big)\psi_Z(w), \quad |w| < q^{-1}.$$

Let $\bar{\gamma}_j = P(Z=j)$ be the PMF of Z. Then, using definition of the PGF, we have

$$\sum_{j=0}^{\infty}(j+1)\bar{\gamma}_{j+1}w^j = \lambda\sum_{j=0}^{\infty}\bar{\gamma}_j w^j + \sum_{j=0}^{\infty} q^{j+1}w^j \sum_{k=0}^{\infty}\bar{\gamma}_k w^k.$$

This implies

$$\sum_{j=0}^{\infty}(j+1)\bar{\gamma}_{j+1}w^j - \lambda\sum_{j=0}^{\infty}\bar{\gamma}_j w^j - \sum_{j=0}^{\infty}\left(\sum_{k=0}^{j}\bar{\gamma}_k q^{j-k+1}\right)w^j = 0.$$

Collecting the coefficients of w^j, we get

$$(j+1)\bar{\gamma}_{j+1} - \lambda\bar{\gamma}_j - \sum_{k=0}^{j}\bar{\gamma}_k q^{j-k+1} = 0.$$

Let $h \in H_Z$ as defined in (5), then

$$\sum_{j=0}^{\infty} h(j+1)\Big[\lambda\bar{\gamma}_j - (j+1)\bar{\gamma}_{j+1} + \sum_{k=0}^{j}\bar{\gamma}_k q^{j-k+1}\Big] = 0.$$

Further simplification leads to

$$\sum_{j=0}^{\infty}\Big[\lambda h(j+1) - jh(j) + \sum_{k=0}^{\infty} q^{k+1}h(j+k+1)\Big]\bar{\gamma}_j = 0.$$

Therefore,

$$\bar{\mathcal{A}}_{X+Y}h(j) = \lambda h(j+1) - jh(j) + \sum_{k=0}^{\infty} q^{k+1}h(j+k+1).$$

Using (13), the proof follows. $\square$

4 Approximation Results

In this section, we derive an error bound for the Poisson and Poisson convoluted geometric approximation to W_n. The following theorem gives the bound for Poisson, with parameter μ, approximation.

Theorem 4.1. *Let $\xi_1, \xi_2, \dots, \xi_n$ be independent random variables satisfying* (3) *and $W_n = \sum_{i=1}^{n} \xi_i$. Then*

$$d_{TV}(W_n, X) \leq \frac{|\mu_2|}{\max(1, \mu)},$$

where $X \sim Po(\mu)$.

Proof. From Proposition 3.1, a Stein operator for W_n is given by

$$\mathcal{A}_{W_n} h(j) = \mu h(j+1) - j h(j) + \sum_{i=1}^{n} \sum_{k=0}^{\infty} \sum_{l=1}^{k} g_{i,k+1} \Delta h(j+l) = \mathcal{A}_X h(j) + \mathcal{U}_{W_n} h(j),$$

where $\mathcal{A}_X$ is a Stein operator for X as discussed in (8). Observe that $\mathcal{A}_{W_n}$ is a Stein operator for W_n which can be seen as a perturbation of Poisson operator. Now, for $h \in H_X \cap H_{W_n}$, taking expectation of perturbed operator $\mathcal{U}_{W_n}$ with respect to W_n and using (9), the result follows. $\square$

Next, we derive $Z = X + Y$ approximation to W_n, where $X \sim \mathrm{Po}(\lambda)$ and $Y \sim \mathrm{Ge}(p)$, by matching first two moments, that is, $\mathbb{E}(Z) = \mathbb{E}(W_n)$ and $\mathrm{Var}(Z) = \mathrm{Var}(W_n)$ which give the following choice of parameters

$$\lambda = \mu - \sqrt{\sigma^2 - \mu} \quad \text{and} \quad p = \frac{1}{1 + \sqrt{\sigma^2 - \mu}}. \tag{14}$$

Theorem 4.2. *Let $\xi_1, \xi_2, \dots, \xi_n$ be independent random variables satisfying* (3) *and the mean and variance of $W_n = \sum_{i=1}^{n} \xi_i$ satisfying* (14)*. Also, assume that $\sigma^2 > \mu$ and $\lambda > 2(q/p)^2$. Then*

$$d_{TV}(W_n, Z) \leq \frac{\lambda \sqrt{\frac{2}{\pi}} \left| \mu_3 - 2\,(q/p)^3 \right| \left(\frac{1}{4} + \sum_{i=1}^{n} \left(1 - d_{TV}(\xi_i, \xi_i + 1)\right)\right)^{-1/2}}{\left(\lambda - 2(q/p)^2\right) \max(1, \lambda)},$$

where $Z = X + Y$, $X \sim Po(\lambda)$ and $Y \sim Ge(p)$.

Remark 4.1. *Note that, in Theorem 4.2, the choice of parameters are valid as*

$$\mu = \lambda + \frac{q}{p} > \frac{q}{p} = \sqrt{\sigma^2 - \mu} \quad \textit{and} \quad p = \frac{1}{1 + \sqrt{\sigma^2 - \mu}} \leq 1,$$

since $\sigma^2 > \mu$.

Proof of Theorem 4.2. From (12), the Stein operator for W_n is given by

$$\mathcal{A}_{W_n}h(j) = \sum_{i=1}^{n}\sum_{k=0}^{\infty} g_{i,k+1}h(j+k+1) - jh(j).$$

Using (13), with $\sum_{i=1}^{n}\sum_{k=0}^{\infty} g_{i,k+1} = \mathbb{E}(W_n) = \mathbb{E}(Z) = \lambda + q/p$, we get

$$\begin{aligned}
\mathcal{A}_{W_n}h(j) &= \Big(\lambda + \frac{q}{p}\Big)h(j+1) - jh(j) + \sum_{k=0}^{\infty}\sum_{l=1}^{k} q^{k+1}\Delta h(j+l) \\
&\quad + \sum_{i=1}^{n}\sum_{k=0}^{\infty}\sum_{l=1}^{k} g_{i,k+1}\Delta h(j+l) - \sum_{k=0}^{\infty}\sum_{l=1}^{k} q^{k+1}\Delta h(j+l) \\
&= \mathcal{A}_Z h(j) + \bar{\mathcal{U}}_{W_n}h(j).
\end{aligned}$$

This is a Stein operator for W_n which can be seen as perturbation of $Z = X + Y$ operator, obtained in Proposition 3.2. Now, consider

$$\bar{\mathcal{U}}_{W_n}h(j) = \sum_{i=1}^{n}\sum_{k=0}^{\infty}\sum_{l=1}^{k} g_{i,k+1}\Delta h(j+l) - \sum_{k=0}^{\infty}\sum_{l=1}^{k} q^{k+1}\Delta h(j+l). \tag{15}$$

We know that

$$\Delta h(j+l) = \sum_{m=1}^{l-1}\Delta^2 h(j+m) + \Delta h(j+1).$$

Substituting in (15) and using $\mathrm{Var}(Z) = \mathrm{Var}(W_n)$ with $\sum_{i=1}^{n}\sum_{k=0}^{\infty} g_{i,k+1} = \mathbb{E}(W_n) = \mathbb{E}(Z) = \lambda + q/p$, we have

$$\bar{\mathcal{U}}_{W_n}h(j) = \sum_{i=1}^{n}\sum_{k=0}^{\infty}\sum_{l=1}^{k}\sum_{m=1}^{l-1} g_{i,k+1}\Delta^2 h(j+m) - \sum_{k=0}^{\infty}\sum_{l=1}^{k}\sum_{m=1}^{l-1} q^{k+1}\Delta^2 h(j+m).$$

Now, taking expectation with respect to W_n, we get

$$\begin{aligned}
\mathbb{E}\big[\bar{\mathcal{U}}_{W_n}h(W_n)\big] &= \sum_{j=0}^{\infty}\Big[\sum_{i=1}^{n}\sum_{k=0}^{\infty}\sum_{l=1}^{k}\sum_{m=1}^{l-1} g_{i,k+1}\Delta^2 h(j+m) \\
&\quad - \sum_{k=0}^{\infty}\sum_{l=1}^{k}\sum_{m=1}^{l-1} q^{k+1}\Delta^2 h(j+m)\Big] P[W_n = j].
\end{aligned}$$

Therefore,

$$\begin{aligned}
\big|\mathbb{E}\big[\bar{\mathcal{U}}_{W_n}h(W_n)\big]\big| &\leq 2d_{TV}(W_n, W_n+1)\|\Delta h\| \left|\sum_{i=1}^{n}\sum_{k=0}^{\infty}\frac{k(k-1)}{2}g_{i,k+1} - \sum_{k=0}^{\infty}\frac{k(k-1)}{2}q^{k+1}\right|. \\
&\leq d_{TV}(W_n, W_n+1)\|\Delta h\| \left|\mu_3 - 2\frac{q^3}{p^3}\right|.
\end{aligned}$$

Using (11), we have

$$\left|\mathbb{E}\left[\mathcal{U}_{W_n}h(W_n)\right]\right| \leq \|\Delta h\|\sqrt{\frac{2}{\pi}}\left(\frac{1}{4}+\sum_{i=1}^{n}\left(1-d_{TV}(\xi_i,\xi_i+1)\right)\right)^{-1/2}\left|\mu_3-2\frac{q^3}{p^3}\right|. \tag{16}$$

From Proposition 3.2, we have

$$\|\mathcal{U}_{X+Y}h\| \leq \frac{q^2}{p^2}\|\Delta h\|. \tag{17}$$

Using (10), (16), and (17) with Lemma 2.1, the proof follows. $\square$

5 An Application to the Waiting Time Distribution of 2-runs

The concept of runs and patterns is well-known in the literature due to its applicability in many real-life applications such as reliability theory, machine maintenance, statistical testing, and quality control, among many others. In this section, we consider the set up discussed by Hirano (1984) and generalized by Huang and Tsai (1991) as follows:
Let N denote the number of two consecutive successes in n Bernoulli trials with success probability p. Then, Huang and Tsai (1991) (with $k_1 = 0$ and $k_2 = 2$ in their notation) have shown that the waiting time for nth occurrence of 2-runs can be written as the sum of n independent and identical distributed (iid) random variables, say $U_1, U_2, \dots, U_n$, concentrated on $\{2, 3, \dots\}$. Here U_i is 2 plus the number of trials between the $(j-1)$th and jth occurrence of 2-runs. The PGF of U_i is given by

$$\psi_U(t) = \frac{p^2t^2}{1-t+p^2t^2},$$

where U is the iid copy of U_i, $i = 1, 2, \dots, n$ (see Hung and Tsai (1991) for more details). Now, let $V_i = U_i - 2$ concentrated on $\mathbb{Z}_+$. Then, Kumar and Upadhye (2017) have given the PGF of V_i and which is given by

$$\psi_{V_i}(t) = \frac{p^2}{1-t+p^2t^2} = \sum_{j=0}^{\infty}\left(\sum_{\ell=0}^{\lfloor j/2\rfloor}\binom{j-\ell}{\ell}(-1)^{\ell}p^{2(\ell+1)}\right)t^j = \sum_{j=0}^{\infty} g_{i,j+1}t^j,$$

where $g_{i,j+1} = \sum_{\ell=0}^{\lfloor j/2\rfloor}\binom{j-\ell}{\ell}(-1)^{\ell}p^{2(\ell+1)}$, for each $i = 1, 2, \dots, n$. For more details, we refer the reader to Huang and Tsai (1991), Kumar and Upadhye (2017), and Balakrishnan and Koutras (2002), and references therein.
Now, let $W_{\bar{n}} = \sum_{i=1}^{\bar{n}} V_i$ then $W_{\bar{n}}$ denotes the number of failures before $\bar{n}^{\text{th}}$ occurrence of 2-runs. Therefore, from Theorem 4.1, we have

$$d_{TV}(W_{\bar{n}}, Po(\mu)) \leq \frac{|\mu_2|}{\max(1,\mu)},$$

where $\mu = \bar{n}\sum_{j=0}^{\infty} g_{i,j+1}$ and $\mu_2 = \bar{n}\sum_{j=0}^{\infty} jg_{i,j+1}$. In a similar manner, from Theorem 4.2, we can also obtain the bound for the Poisson convoluted geometric approximation. For more details, we refer the reader to Section 4 of Kumar and Upadhye (2017).

6 Concluding Remarks

1. Note that, if $\xi_i \sim Po(\lambda_i)$, $i = 1, 2, \ldots, n$ then $d_{TV}(W_n, X) = 0$ in Theorem 4.1, as expected.

2. If $\xi_1 \sim Po(\lambda)$ and $\xi_2 \sim Ge(p)$, for $i = 1, 2$, and $W_2 = \xi_1 + \xi_2$ then $d_{TV}(W_2, Z) = 0$ in Theorem 4.2, as expected.

3. The bounds obtained in Theorems 4.1 and 4.2 are either comparable to or improvement over the existing bounds available in the literature. In particular, some comparison can be seen as follows:

   (a) If $\xi_i \sim Ber(p_i)$, for $i = 1, 2, \ldots, n$ then, from Theorem 4.1, we have

   $$d_{TV}(W_n, Po(\mu)) \leq \frac{1}{\max(1, \mu)} \sum_{i=1}^{n} p_i^2,$$

   where $\mu = \sum_{i=1}^{n} p_i$. The above bound is same as given by Barbour *et al.* (1992b) and is an improvement over the bound $d_{TV}(W_n, Po(\mu)) \leq \sum_{i=1}^{n} p_i^2$ given by Khintchine (1933) and Le Cam (1960).

   (b) If $\xi_i \sim Ge(p_i)$, $i = 1, 2, \ldots, n$ then, from Theorem 4.1, we have

   $$d_{TV}(W_n, X) \leq \frac{1}{\max(1, \mu)} \sum_{i=1}^{n} \left(\frac{q_i}{p_i}\right)^2.$$

   This bound is an improvement over negative binomial approximation given by Kumar and Upadhye (2017) in Corollary 3.1.

   (c) If $\xi_i \sim NB(\alpha_i, p_i)$, $i = 1, 2, \ldots, n$ then, from Theorems 4.1, we have

   $$d_{TV}(W_n, Po(\mu)) \leq \frac{1}{\max(1, \mu)} \sum_{i=1}^{n} \alpha_i \left(\frac{q_i}{p_i}\right)^2, \tag{18}$$

   where $\mu = \sum_{i=1}^{n} \frac{\alpha_i q_i}{p_i}$. Vellaisamy and Upadhye (2009) obtained bound for $S_n = \sum_{i=1}^{n} \xi_i$ and is given by

   $$d_{TV}(S_n, Po(\lambda)) \leq \min\left(1, \frac{1}{\sqrt{2\lambda e}}\right) \sum_{i=1}^{n} \frac{\alpha_i q_i^2}{p_i}, \tag{19}$$

   where $\lambda = \sum_{i=1}^{n} \alpha_i q_i = \alpha q$. Under identical set up with $\alpha = 5$ and various values of n and q, the numerical comparison of (18) and (19) as follows:

Table 1: Comparison of bounds.

n	q	From (18)	From (19)
10		0.1111	0.3370
30	0.1	0.1111	1.0109
50		0.1111	1.6848
10		0.2500	1.0722
30	0.2	0.2500	3.2166
50		0.2500	5.3610

Note that our bound (from (18)) is better than the bound given in (19). In particular, graphically, the closeness of these two distributions can be seen as follows:

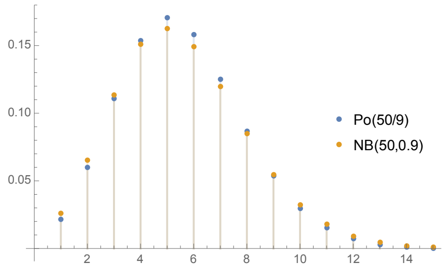

Figure 1: $n = 10, q = 0.1$

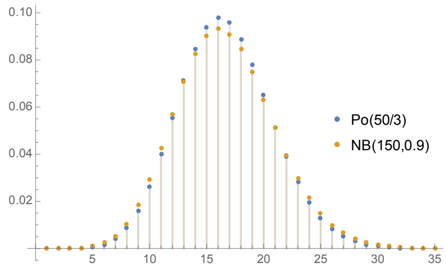

Figure 2: $n = 30, q = 0.1$

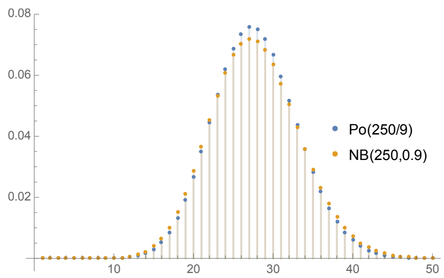

Figure 3: $n = 50, q = 0.1$

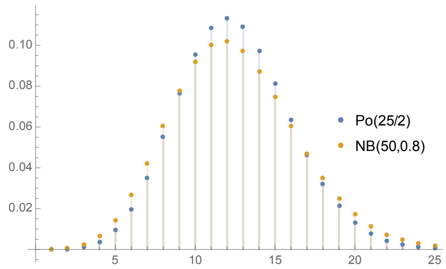

Figure 4: $n = 10, q = 0.2$

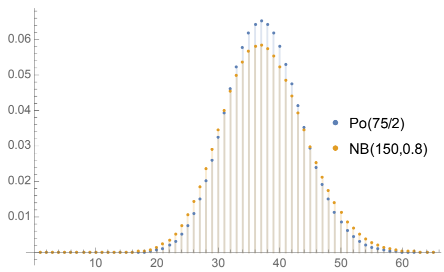

Figure 5: $n = 30, q = 0.2$

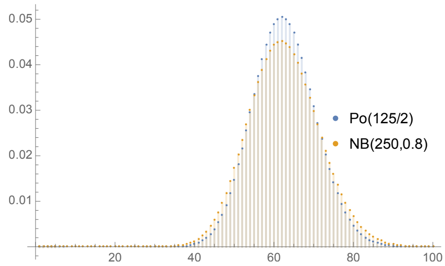

Figure 6: $n = 50, q = 0.2$

The above graphs are obtained by using the moment matching conditions. Also, from the numerical table and graphs, observe that the distributions are closer for sufficiently small values of q and large values of n, as expected.

(d) From Theorem 1 of Hung and Giang (2016), it is given that, for $A \subset \mathbb{Z}_+$,

$$\sup_A \left| P(W_n \in A) - \sum_{k \in A} \frac{\lambda_n^k e^{-\lambda_n}}{k!} \right|$$
$$\leq \sum_{i=1}^{n} \min\left\{ \lambda_n^{-1}(1 - e^{-\lambda_n}) r_{n,i}(1 - p_{n,i}), 1 - p_{n,i} \right\} (1 - p_{n,i}) p_{n,i}^{-1}, \qquad (20)$$

where $W_n = \sum_{i=1}^{n} X_{n,i}$, $X_{n,i} \sim NB(r_{n,i}, p_{n,i})$ with $\lambda_n = \mathbb{E}(W_n)$. Note that if $\min\left\{\lambda_n^{-1}(1 - e^{-\lambda_n}) r_{n,i}(1 - p_{n,i}), 1 - p_{n,i}\right\} = 1 - p_{n,i}$, for all $i = 1, 2, \ldots, n$, then

$$\sup_A \left| P(W_n \in A) - \sum_{k \in A} \frac{\lambda_n^k e^{-\lambda_n}}{k!} \right| \leq \sum_{i=1}^{n} (1 - p_{n,i})^2 p_{n,i}^{-1}, \qquad (21)$$

which is of order $O(n)$. Clearly, for large values of n, Theorem 4.1 is an improvement over (21).

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