Dynamic reliability modeling with medical applications

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Abstract:

- This article explores a dynamic reliability index for dependent stress and strength variables. Toward this end, a copula approach is utilized to model the association between the two variables. Under Farlie-Gumbel-Morgenstern copula and its generalization, expressions of the reliability measure are derived. Numerical results are used to assess effect of the marginal distributions and the reference copula parameters on the reliability index. Application of the proposed method in the context of medicine is also presented.

Key-Words:

- *Copula; Dependence structure; Stress-strength model.*

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1. INTRODUCTION

Let X and Y be two continuous random variables. A large body of literature has grown around statistical inference for $R = P(X > Y)$. This enthusiasm roots in applicability of this quantity in diverse areas. In the so-called stress-strength model in engineering, R measures the reliability of a component, where X and Y represent the strength of the component, and the stress that it is undergoing, respectively. For example, Weerahandi and Johnson [22] considered a rocket-motor experiment in which X represents the chamber burst strength, and Y represents the operating pressure. In medicine, R may be interpreted as a measure of treatment's effectiveness if X and Y are the response variables from treatment and control groups, respectively (Ventura and Racugno [20]). It is also related to receiver operating characteristic (ROC) curve, which is a useful tool in analysis of the discriminatory accuracy of a diagnostic test or marker in distinguishing between diseased and non-diseased individuals. Bamber [2] showed that the area under the ROC curve equals R. Wolfe and Hogg [23] considered R as a general measure for the difference between two populations.

The estimation of R has received considerable attention in the statistical literature. A comprehensive account of this topic appears in Kotz et al. [11]. To facilitate mathematical development, most of the pertinent articles assume that X and Y are independent. In many real situations, however, the two variables are correlated. In the following, three examples in the context of engineering, education and economics are presented (see Domma and Giordano [4]):

- Let X and Y be the lifetimes of two electronic devices, stimulated by a single source. Then R is the probability that one survives after the other one.
- Some universities in Japan use an admission test based on Japanese (X) and English (E) knowledge. In order to get admission, a candidate must qualify $X + E > c$, where c is a pre-determined cut-off score. If we set $Y = c - E$, then the admission probability is given by R.
- Let X and Y be household consumption and income, respectively. If consumption exceeds income, then household will face financial stress. Thus, R is a measure of household financial fragility.

The reliability estimation has been studied for some bivariate distributions, including bivariate normal (Gupta and Subramanian [10]), bivariate beta (Nadarajah [15]), bivariate exponential (Nadarajah and Kotz [16]), and bivariate log-normal (Gupta et al. [9]), among others. A limitation shared by these articles is that the marginal distributions are of the same type. Moreover, a specific form of dependence between margins is allowed. Bivariate normal distribution is a nice example clarifying these points. Here, the marginal distributions are normal, and their association is linear. To overcome the above shortcomings, Domma and Giordano [4] built on a copula to model the association between the two variables.

Let the random variables X and Y be the lifetimes of two systems. If both systems are operating at time $t > 0$, then their residual lifetimes are given by $X_t = (X - t|X > t)$ and $Y_t = (Y - t|Y > t)$. Zardasht and Asadi [24] proposed $R(t) = P(X_t > Y_t)$ as a time-dependent criterion to compare the two residual lifetimes. They studied properties of this measure, and developed a nonparametric estimator for $R(t)$ based on two independent random samples. Mahdizadeh and Zamanzade [12, 13, 14] are examples of recent works on inference about $R(t)$. In light of the above argument for R, we think that $R(t)$ is also applicable in settings where X_t and Y_t are not independent. For example, in the third example provided above, $R(t)$ can be considered as a measure of household financial fragility, given that the consumption and income exceed a lower bound t. This article employs a copula approach to account for dependence in evaluating $R(t)$. Our approach is similar to that adopted by Domma and Giordano [4].

Section 2 presents some basic properties of copulas. Section 3 provides expressions of $R(t)$ for some parametric family of copulas, and margins. Section 4 contains numerical results evaluating the effect of the marginal distributions and the reference copula parameters on the reliability index. In Section 5, the proposed method is applied to a data set. Final conclusions appear in Section 6. Figures are collected in an appendix.

2. THE COPULA APPROACH

If $\mathcal{I}$ is the interval [0,1], then a bivariate copula can be represented as $C : \mathcal{I} \times \mathcal{I} \to \mathcal{I}$, where C fulfils the following properties:

- For all $u, v \in \mathcal{I}$, $C(u, 0) = 0$, $C(0, v) = 0$, $C(u, 1) = u$, and $C(1, v) = v$.
- For all $u_1, u_2, v_1, v_2 \in \mathcal{I}$, with $u_1 \leq u_2$ and $v_1 \leq v_2$,

$$C(u_2, v_2) - C(u_2, v_1) - C(u_1, v_2) + C(u_1, v_1) \geq 0.$$

A famous theorem by Sklar [19] provides the connection between bivariate copulas and bivariate distribution functions. It states that for any two continuous random variables X and Y with joint distribution function H, there exists a unique copula C such that

$$H(x, y) = C(F(x), G(y)), \quad \forall x, y \in \mathbb{R},$$

where F and G are the marginal distributions of X and Y, respectively. Let f and g be the corresponding marginal density functions. Then, the joint density function is

$$h(x, y) = c(F(x), G(y))f(x)g(y), \tag{2.1}$$

where $c(F(x), G(y)) = \frac{\partial^2 C(F(x),G(y))}{\partial F(x) \partial G(y)}$ is called the copula density.

In general, $C \in C_{\boldsymbol{\theta}}$, where $\boldsymbol{\theta}$ is a vector of parameters that determines the degree of dependence between the two random variables. Also, $F \in F_{\boldsymbol{\gamma}}$ and $G \in G_{\boldsymbol{\nu}}$, where $\boldsymbol{\gamma}$ and $\boldsymbol{\nu}$ are vectors of parameters associated with the marginal distributions. For simplicity in notation, all such parameters are assumed implicitly.

A salient feature of copulas is that they allow us to model the dependence structure between random variables independently of the marginal distributions. Owing to this flexibility, the copula approach has drawn much interest in recent years. It has been successfully applied in a variety of scientific fields. Some applications are provided in the following. In biomedical research, Escarela and Carrière [5] employed copula in studying competing risks. In the actuarial context, Frees and Wang [6] modeled dependent mortality and losses using copulas. In the engineering context, Genest and Favre [7] utilized copulas in hydrological modeling.

3. COMPUTATION OF $R(t)$

We first provide a representation for $R(t)$ which is helpful in our mathematical development. It is easily seen that

$$R(t) = \frac{R_1(t)}{R_2(t)}, \tag{3.1}$$

where $R_1(t) = P(X > Y > t)$ and $R_2(t) = P(X > t, Y > t)$. Using $h(x, y)$ in (2.1), components of $R(t)$ can be written as

$$R_1(t) = \int_t^\infty \int_t^x c(F(x), G(y)) f(x) g(y) \, \mathrm{d}y \, \mathrm{d}x, \tag{3.2}$$

and

$$R_2(t) = \int_t^\infty \int_t^\infty c(F(x), G(y)) f(x) g(y) \, \mathrm{d}y \, \mathrm{d}x. \tag{3.3}$$

In the following, the marginal distributions and the copulas used in computing (3.1) are introduced.

Burr [3] introduced a family of distributions that includes twelve distribution types. Two important cases are the Burr type III (BIII), and Burr type XII. The former distribution is more flexible in the sense that it covers wider ranges of skewness and kurtosis, often exhibited by real data. It has been applied in a multitude of data-modeling contexts. The interested reader is referred to Zimmer et al. [25] and Shao [18] for some applications in reliability and environmental studies, among others.

The cumulative distribution function (CDF) and the probability density function (PDF) of the BIII distribution are given by

$$F(x) = \left(1 + x^{-\delta}\right)^{-\alpha}, \quad x > 0;\ \alpha, \delta > 0,$$

and

$$f(x) = \alpha\delta x^{-(\delta+1)} \left(1 + x^{-\delta}\right)^{-(\alpha+1)}, \quad x > 0;\ \alpha, \delta > 0,$$

respectively. The random variable X with this distribution will be denoted by $X \sim \mathrm{BIII}(\alpha, \delta)$. In our reliability modeling, it is assumed that both stress and strength variables follow the BIII distribution. The positivity assumption for X and Y is not restrictive, because it is possible to use an increasing transformation to create positive random variables from arbitrary X and Y, while preserving the dependence structure. See Theorem 2.4.3 in Nelsen [17].

To model the association between the two variables, we consider two famous copulas: Farlie-Gumbel-Morgenstern (FGM), and generalized Farlie-Gumbel-Morgenstern (GFGM). These copulas enjoy the advantage of mathematical tractability. In particular, it turns out that under both families, $R_1(t)$ and $R_2(t)$ are decomposed into two components. The first one represents the numerator/denominator in (3.1) when X and Y are independent, and the second one indicates contribution of the association between the two variables in the value of the corresponding quantity. This property is not shared by all other copulas.

3.1. USING FGM COPULA

The FGM copula is one of the most popular parametric family of copulas that has been widely used due to its simple form. It is defined as

$$C\left(F(x), G(y)\right) = F(x)G(y)\left(1 + \theta\left[1 - F(x)\right]\left[1 - G(y)\right]\right), \quad \theta \in [-1, 1].$$

The corresponding copula density is given by

$$c\left(F(x), G(y)\right) = 1 + \theta\left[1 - 2F(x)\right]\left[1 - 2G(y)\right], \quad \theta \in [-1, 1]. \tag{3.4}$$

Substituting (3.4) in (3.2) and with some algebra, it follows that

$$R_1(t) = R_1^I(t) + \theta R_1^D(t),$$

where

$$R_1^I(t) = \int_t^{\infty} G(x)\,\mathrm{d}F(x) - G(t)\left[1 - F(t)\right], \tag{3.5}$$

and

$$R_1^D(t) = \int_t^{\infty} \left[1 - 2F(x)\right]\left[G(x) - G^2(x)\right]\mathrm{d}F(x) + \left[F(t) - F^2(t)\right]\left[G(t) - G^2(t)\right]. \tag{3.6}$$

Again, substituting (3.4) in (3.3) and some simplification yield

$$R_2(t) = R_2^I(t) + \theta R_2^D(t),$$

where

$$R_2^I(t) = \left[1 - F(t)\right]\left[1 - G(t)\right], \tag{3.7}$$

and

$$R_2^D(t) = F(t)G(t)\left[1 - F(t)\right]\left[1 - G(t)\right]. \tag{3.8}$$

If $X \sim \mathrm{BIII}(\alpha, \delta)$ and $Y \sim \mathrm{BIII}(\beta, \delta)$, then it is possible to obtain a closed-form expression for $R(t)$. For notational convenience, $S(t; \delta, k)$ is defined as

$$S(t; \delta, k) = \frac{1}{k}\left[1 - \left(1 + t^{-\delta}\right)^{-k}\right], \quad t, k, \delta > 0. \tag{3.9}$$

Incorporating the PDF and CDF of the the BIII distribution in (3.5) and (3.6), we get

$$R_1^I(t) = \alpha\, S(t; \delta, \alpha + \beta) - \alpha\, S(t; \delta, \alpha)\Big[1 - \beta\, S(t; \delta, \beta)\Big],$$

and

$$\begin{aligned} R_1^D(t) \;=\; & \alpha\Big[S(t; \delta, \alpha + \beta) - S(t; \delta, \alpha + 2\beta)\Big] \\ & - 2\alpha\Big[S(t; \delta, 2\alpha + \beta) - S(t; \delta, 2(\alpha + \beta))\Big] \\ & + \alpha\,\beta\, S(t; \delta, \alpha)\, S(t; \delta, \beta)\Big[1 - \alpha\, S(t; \delta, \alpha)\Big]\Big[1 - \beta\, S(t; \delta, \beta)\Big]. \end{aligned}$$

Similarly, one can verify that (3.7) and (3.8) take the forms

$$R_2^I(t) = \alpha\,\beta\, S(t; \delta, \alpha)\, S(t; \delta, \beta),$$

and

$$R_2^D(t) = \alpha\,\beta\, S(t; \delta, \alpha)\, S(t; \delta, \beta)\Big[1 - \alpha\, S(t; \delta, \alpha)\Big]\Big[1 - \beta\, S(t; \delta, \beta)\Big].$$

3.2. USING GFGM COPULA

Any copula depends on some parameters which determine the degree of dependence between the margins. Two common measures of the association are Spearman's ρ and Kendall's τ coefficients. Under the FGM copula, $\rho \in [-1/3, 1/3]$ and $\tau \in [-2/9, 2/9]$, meaning that a relatively weak dependence is allowed. As a result, several modifications of the original FGM copula have been proposed. In the following, we consider a generalization due to Bairamov et al. [1]. The GFGM copula is defined as

$$C\left(F(x),G(y)\right)=F(x)G(y)\left(1+\theta\left[1-F(x)^{m_1}\right]^{p_1}\left[1-G(y)^{m_2}\right]^{p_2}\right),$$

where m_1, m_2, p_1, and p_2 are positive parameters, and $\theta\in[\theta_\ell,\theta_u]$ with

$$\theta_\ell=-\min\left\{1,\frac{1}{m_1m_2}\left(\frac{1+m_1p_1}{m_1(p_1-1)}\right)^{p_1-1}\left(\frac{1+m_2p_2}{m_2(p_2-1)}\right)^{p_2-1}\right\},$$

and

$$\theta_u=\min\left\{\frac{1}{m_1}\left(\frac{1+m_1p_1}{m_1(p_1-1)}\right)^{p_1-1},\frac{1}{m_2}\left(\frac{1+m_2p_2}{m_2(p_2-1)}\right)^{p_2-1}\right\}.$$

The corresponding copula density is given by

$$\begin{aligned}c\left(F(x),G(y)\right)&=1+\theta\left[1-F(x)^{m_1}\right]^{p_1-1}\left[1-(1+m_1p_1)F(x)^{m_1}\right]\\&\quad\times\left[1-G(y)^{m_2}\right]^{p_2-1}\left[1-(1+m_2p_2)G(y)^{m_2}\right].\end{aligned}\tag{3.10}$$

Clearly, by setting $m_1=m_2=p_1=p_2=1$ in the above equation, we arrive at (3.4).

Let p_1 and p_2 be two positive integers. Then using the binomial expansion in (3.10), it can be shown that

$$\begin{aligned}c\left(F(x),G(y)\right)&=1+\theta\sum_{i=0}^{p_1-1}\sum_{j=0}^{p_2-1}\binom{p_1-1}{i}\binom{p_2-1}{j}(-1)^{i+j}F(x)^{m_1i}G(y)^{m_2j}\\&\quad\times\left[1-(1+m_1p_1)F(x)^{m_1}\right]\left[1-(1+m_2p_2)G(y)^{m_2}\right].\end{aligned}\tag{3.11}$$

This representation will be used in computing $R(t)$. Proceeding as in the previous sub-section, we get

$$R_1(t)=R_1^I(t)+\theta R_1^D(t),$$

where $R_1^I(t)$ is given in (3.5), and

$$R_1^D(t)=\sum_{i=0}^{p_1-1}\binom{p_1-1}{i}(-1)^i\int_t^\infty F(x)^{m_1i}\left[1-(1+m_1p_1)F(x)^{m_1}\right]J(x)\,\mathrm{d}F(x),\tag{3.12}$$

with

$$\begin{aligned}J(x)&=\sum_{j=0}^{p_2-1}\binom{p_2-1}{j}(-1)^j\Bigg(\frac{1}{m_2j+1}\left[G(x)^{m_2j+1}-G(t)^{m_2j+1}\right]\\&\quad-\frac{1+m_2p_2}{m_2(j+1)+1}\left[G(x)^{m_2(j+1)+1}-G(t)^{m_2(j+1)+1}\right]\Bigg).\end{aligned}$$

Similarly, it is concluded that

$$R_2(t)=R_2^I(t)+\theta R_2^D(t),$$

where $R_2^I(t)$ is given in (3.7), and

$$\begin{aligned} R_2^D(t) \;=\; & \sum_{i=0}^{p_1-1}\sum_{j=0}^{p_2-1}\binom{p_1-1}{i}\binom{p_2-1}{j}(-1)^{i+j} \\ & \times\left(\frac{1}{m_1 i+1}\left[1-F(t)^{m_1 i+1}\right]-\frac{1+m_1p_1}{m_1(i+1)+1}\left[1-F(t)^{m_1(i+1)+1}\right]\right) \\ & \times\left(\frac{1}{m_2 j+1}\left[1-G(t)^{m_2 j+1}\right]-\frac{1+m_2p_2}{m_2(j+1)+1}\left[1-G(t)^{m_2(j+1)+1}\right]\right). \end{aligned} \tag{3.13}$$

If $X \sim \mathrm{BIII}(\alpha,\delta)$, $Y \sim \mathrm{BIII}(\beta,\delta)$, and $S(t;\delta,k)$ is defined as in (3.9), then from (3.12) and (3.13) we have

$$\begin{aligned} R_1^D(t) \;=\; & \sum_{i=0}^{p_1-1}\sum_{j=0}^{p_2-1}\binom{p_1-1}{i}\binom{p_2-1}{j}(-1)^{i+j} \\ & \times\Bigg\{\frac{\alpha}{(m_2 j+1)}\,S(t;\delta,\alpha(m_1 i+1)+\beta(m_2 j+1)) \\ & -\frac{\alpha(1+m_2p_2)}{m_2(j+1)+1}\,S(t;\delta,\alpha(m_1 i+1)+\beta\,(m_2(j+1)+1)) \\ & -\frac{\alpha(1+m_1p_1)}{m_2 j+1}\,S(t;\delta,\alpha\,(m_1(i+1)+1)+\beta(m_2 j+1)) \\ & +\frac{\alpha(1+m_1p_1)(1+m_2p_2)}{m_2(j+1)+1}\,S(t;\delta,\alpha\,(m_1(i+1)+1)+\beta\,(m_2(j+1)+1)) \\ & +\Bigg((1+m_2p_2)\left[\frac{1}{m_2(j+1)+1}-\beta\,S(t;\delta,\beta\,(m_2(j+1)+1))\right] \\ & -\left[\frac{1}{m_2 j+1}-\beta\,S(t;\delta,\beta(m_2 j+1))\right]\Bigg) \\ & \times\alpha\Big[S(t;\delta,\alpha(m_1 i+1))-(1+m_1p_1)\,S(t;\delta,\alpha\,(m_1(i+1)+1))\Big]\Bigg\}, \end{aligned}$$

and

$$\begin{aligned} R_2^D(t) \;=\; & \sum_{i=0}^{p_1-1}\sum_{j=0}^{p_2-1}\binom{p_1-1}{i}\binom{p_2-1}{j}(-1)^{i+j} \\ & \times\alpha\Big[S(t;\delta,\alpha(m_1 i+1))-(1+m_1p_1)\,S(t;\delta,\alpha\,(m_1(i+1)+1))\Big] \\ & \times\beta\Big[S(t;\delta,\beta(m_2 j+1))-(1+m_2p_2)\,S(t;\delta,\beta\,(m_2(j+1)+1))\Big]. \end{aligned}$$

4. NUMERICAL RESULTS

We now evaluate $R(t)$ for some specific choices of the marginal distributions, and the reference copula parameters. Figures 1-5 show the curves of $R(t)$, where the involved parameters are given in the caption of each figure. The following configurations of

$$(m_1, m_2, p_1, p_2) \in \Big\{(1,1,1,1), (1,4,2,7), (1,4,1,10), (4,1,3,2), (5,5,2,1)\Big\}$$

are associated with Figures 1-5, respectively. In each case, sixteen combinations of (α, β, δ) and θ are considered whose values can be found in the caption of the figures. In particular, black/solid curves indicate the situation that X and Y are independent, i.e. $\theta = 0$ in (3.4) and (3.10).

Figure 1 is given to the FGM copula. For fixed t, $R(t)$ is a decreasing function of θ if $\alpha < \beta$. The situation is reversed if $\alpha > \beta$. For example, compare panels (a) and (c). These properties are easily concluded in the special case of $t = 0$, as mentioned by Domma and Giordano [4]. The plot presented in panel (d) is very interesting. In fact, it can be shown that if the marginal distributions are identical ($\alpha = \beta$), then $R(t) = 0.5$ for all t, regardless of θ. Finally, one can see that $R(t)$ is a monotone function of t, given a fixed θ.

Figures 2-5 correspond to the GFGM copula. Depending on values of the involved parameters, the reliability measure takes a variety of functional forms. A marked difference from Figure 1 is that for fixed θ, $R(t)$ may not be a monotone function of t. This is observed in panel (d) of Figure 3, for example. If the margins are the same and $\theta = 0$ (see Figures 2 and 4), then it can be proved that $R(t) = 0.5$ for all t. We note that under the FGM copula, this property holds for arbitrary θ.

It should be emphasized that if the dependence between X and Y is not incorporated in computing the reliability, the resulting value could be higher/lower than the true one. Compare black/solid curve with the others in each panel of Figures 1-5. This highlights importance of the copula approach as it is an efficient way to capture dependence structure between random variables in developing inference procedures.

5. APPLICATION

In this section, application of the copula-based approach in reliability modeling is provided based on China Health and Nutrition Survey (CHNS) data. The CHNS is an international collaborative project between the Carolina Population Center at the University of North Carolina at Chapel Hill, and the National Institute for Nutrition and Health at the Chinese Center for Disease Control and Prevention. It is designed to examine the effects of the health, nutrition, and family planning policies and programs implemented by national and local governments and to see how the social and economic transformation of Chinese society is affecting the health and nutritional status of its population.

Recent studies support the importance of the lipid-transporting apolipoproteins, such as ApoA and ApoB which transport high-density lipoprotein (HDL, good) cholesterol and low-density lipoprotein (LDL, bad) cholesterol particles, respectively. A healthy individual probably has larger ApoA value than ApoB, and thereby less risk for cardiovascular disease. As alternatives to the traditional LDL and HDL biomarkers, these apolipoproteins have some advantages (Walldius and Jungner [21]). Let R be the probability of ApoA being greater than ApoB, where both were from the same individual, i.e., $R = P(\text{ApoA} > \text{ApoB})$. If this probability is significantly larger than 0.5, then ApoA is stochastically larger than ApoB for the population, meaning that this population is relatively at lower risk of cardiovascular disease. Suppose from the previous studies, the researcher knows a lower bound t for values of the biomarkers in the population. Then, one can utilize the index

$$R(t) = \frac{P(\text{ApoA} > \text{ApoB} > t)}{P(\text{ApoA} > t, \text{ApoB} > t)}.$$

The CHNS data set[1] contains values of ApoA and ApoB biomarkers for 10,187 Chinese children and adults (aged ≥ 7) in year 2009. For the purpose of illustration, we estimated $R(t)$ based on the first 1,000 pairs of data. In doing so, we used the GFGM copula and assumed that the margins are $X \sim \text{BIII}(\alpha, \delta)$ and $Y \sim \text{BIII}(\beta, \delta)$, where X and Y denote ApoA and ApoB, respectively. In particular, the copula parameters were chosen as $m_1 = m_2 = 3$ and $p_1 = p_2 = 2$. This set of values allows for nearly the maximum degree of dependence between the margins under the GFGM copula. Moreover, it simplifies the model through setting $m_1 = m_2$ and $p_1 = p_2$. The last parameter of the copula can be estimated from the expression of Kendall's τ. Domma and Giordano [4] showed that Kendall's τ for the GFGM copula is given by

$$\tau = \frac{8\theta p_1 p_2}{(2 + m_1 p_1)(2 + m_2 p_2)} B\left(\frac{2}{m_1}, p_1\right) B\left(\frac{2}{m_2}, p_2\right),$$

where $B(.,.)$ is the beta function. By replacing τ in the above equation with its value from the sample, and setting $m_1 = m_2 = 3$ and $p_1 = p_2 = 2$, an estimate of θ is obtained as 0.176. It is to be noted that 0.176 falls into admissible range of θ in the GFGM copula with the aforesaid choices of m_i's and p_i's, i.e. [-0.605, 0.778].

Before using the results of sub-section 3.2, it is needed to formally assess fit of the above-mentioned copula to the data. Toward this end, we employed the test statistic $S_n^{(B)}$, introduced by Genest et al. [8], based on Rosenblatt's transform. The P-value associated with this test is determined through parametric

[1] It is available at http://www.cpc.unc.edu/projects/china/data/datasets.

bootstrap, where the details can be found in Appendix D of Genest et al. [8]. In doing so, parameters α, β and δ were estimated from data by 1.286, 0.522 and 7.398, respectively. Based on 1,000 bootstrap replications, an approximate P-value for the test was computed as 0.316. So the null hypothesis that the selected copula fits the data is not rejected at 0.05 significance level. Figure 6 shows the PDF constructed using this specific GFGM copula with the Burr III marginal distributions.

Plugging in the above set of parameters into the expression of $R(t)$ in subsection 3.2 yields an estimate of the dynamic reliability. The corresponding graph is depicted in Figure 7 with blue/dashed curve. A similar graph may be plotted by ignoring the dependence between the two variables, i.e. replacing 0.176 with 0 in the computations. The result is displayed by black/solid curve in Figure 7. It is worth commenting that failing to incorporate the dependence structure leads to inaccuracy in estimating $R(t)$.

6. CONCLUSION

In the classical stress-strength model, the interest centers on $R = P(X > Y)$ for a unit, where X and Y are the strength of the unit and the environmental stress, respectively. This model has attracted much interest in the statistical literature. There are abundant applications in the areas of reliability, quality control, psychology, medicine and clinical trials. Recently, R has been extended to a dynamic form $R(t) = P(X_t > Y_t)$, where X_t and Y_t are residual lifetimes of two systems. Although the latter measure was motivated by a problem in reliability theory, it is potentially applicable in many other situations. This article puts forward a copula approach to account for dependence in evaluating $R(t)$. Some explicit expressions for $R(t)$ are provided when the margins follow the BIII distribution, and the reference copula is either the FGM or GFGM. The proposed method is explored by means of numerical results and real data analysis.

It would be interesting to use other copulas, which allow for higher correlation between the stress and strength variables, in dynamic reliability modeling. This will be considered in a separate study.

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APPENDIX

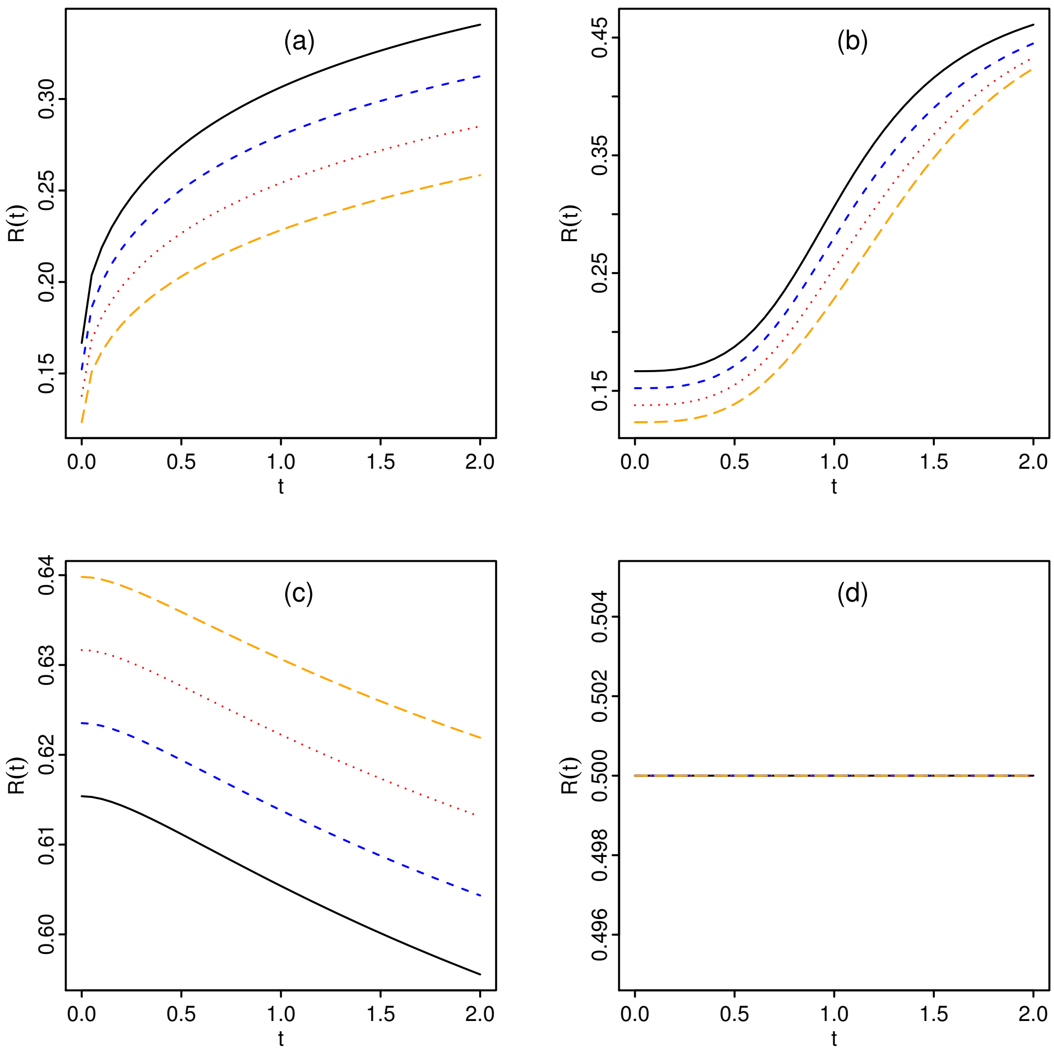

Figure 1: Plot of $R(t)$ based on the FGM copula and the Burr III marginal distributions with: (a) $(\alpha, \beta, \delta) = (1, 5, 0.5)$, (b) $(\alpha, \beta, \delta) = (1, 5, 3)$, (c) $(\alpha, \beta, \delta) = (8, 5, 0.5)$, and (d) $(\alpha, \beta, \delta) = (8, 8, 3)$. Black/solid, blue/dashed, red/dotted, and orange/long-dashed curves relate to $\theta = 0, 0.333, 0.667, 1$, respectively.

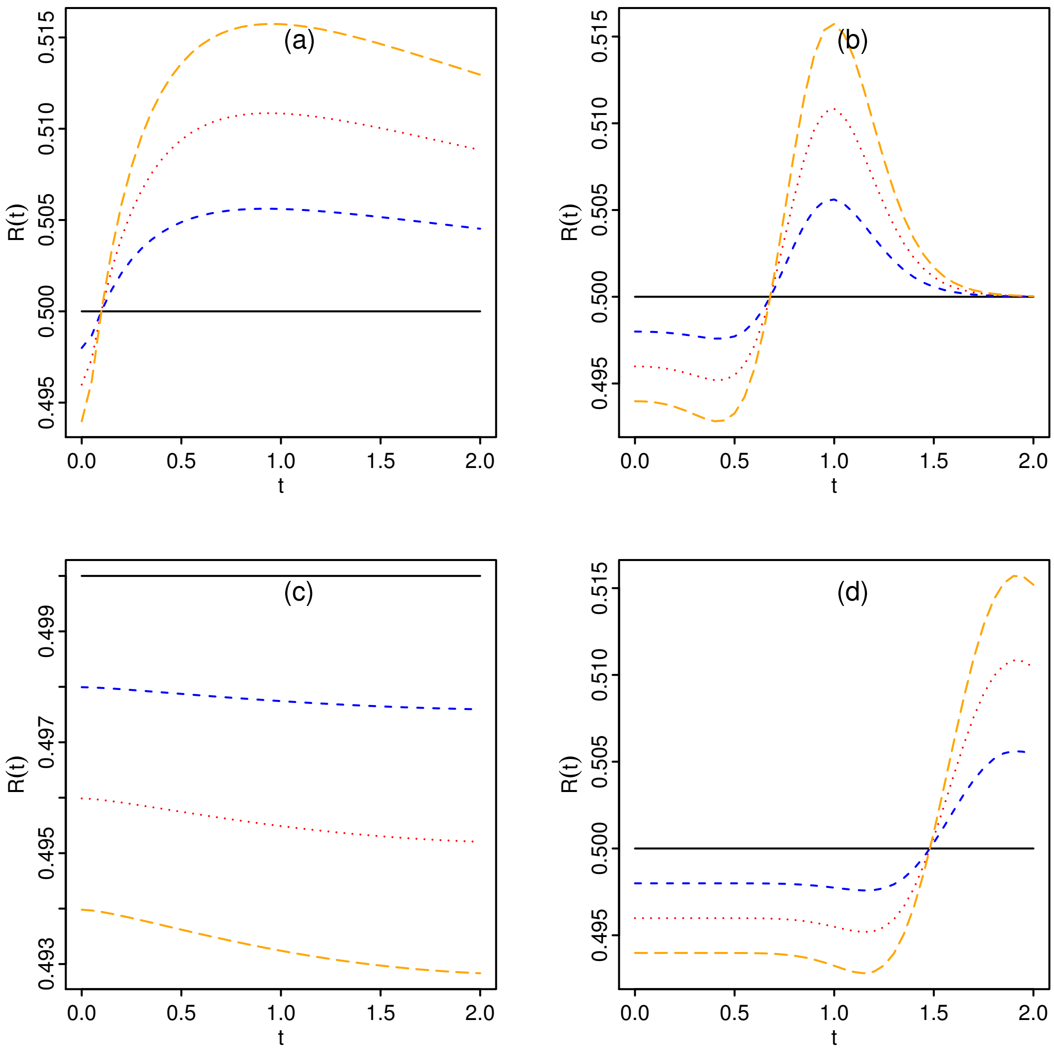

Figure 2: Plot of $R(t)$ based on the GFGM copula with $(m_1, m_2, p_1, p_2) = (1, 4, 2, 7)$, and the Burr III marginal distributions with: (a) $(\alpha, \beta, \delta) = (0.75, 0.75, 0.5)$, (b) $(\alpha, \beta, \delta) = (0.75, 0.75, 3)$, (c) $(\alpha, \beta, \delta) = (4, 4, 0.5)$, and (d) $(\alpha, \beta, \delta) = (4, 4, 3)$. Black/solid, blue/dashed, red/dotted, and orange/long-dashed curves relate to $\theta = 0, 0.259, 0.519, 0.778$, respectively.

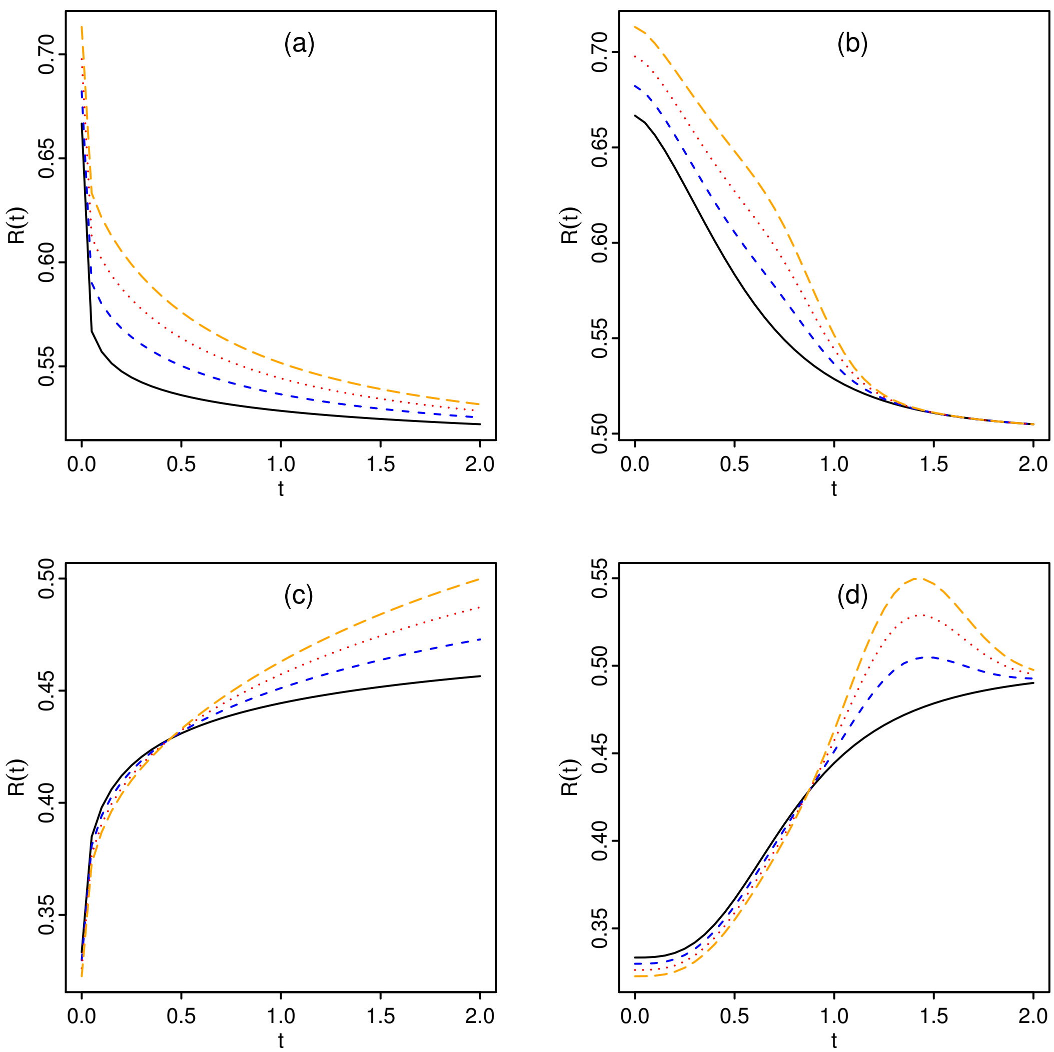

Figure 3: Plot of $R(t)$ based on the GFGM copula with $(m_1, m_2, p_1, p_2) = (1, 4, 1, 10)$, and the Burr III marginal distributions with: (a) $(\alpha, \beta, \delta) = (1, 0.5, 0.5)$, (b) $(\alpha, \beta, \delta) = (1, 0.5, 3)$, (c) $(\alpha, \beta, \delta) = (1, 2, 0.5)$, and (d) $(\alpha, \beta, \delta) = (1, 2, 3)$. Black/solid, blue/dashed, red/dotted, and orange/long-dashed curves relate to $\theta = 0, 0.269, 0.537, 0.806$, respectively.

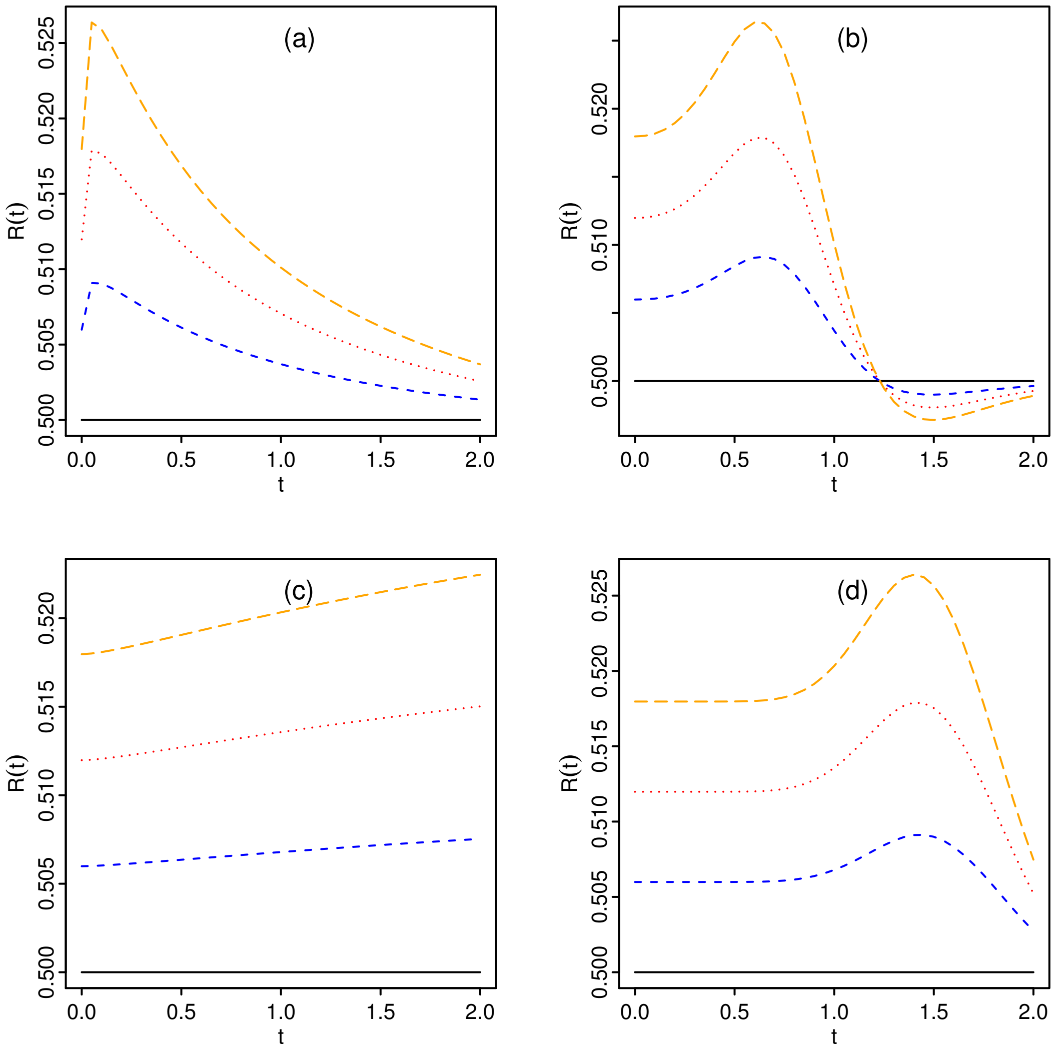

Figure 4: Plot of $R(t)$ based on the GFGM copula with $(m_1, m_2, p_1, p_2) = (4, 1, 3, 2)$, and the Burr III marginal distributions with: (a) $(\alpha, \beta, \delta) = (0.75, 0.75, 0.5)$, (b) $(\alpha, \beta, \delta) = (0.75, 0.75, 3)$, (c) $(\alpha, \beta, \delta) = (4, 4, 0.5)$, and (d) $(\alpha, \beta, \delta) = (4, 4, 3)$. Black/solid, blue/dashed, red/dotted, and orange/long-dashed curves relate to $\theta = 0, 0.22, 0.44, 0.66$, respectively.

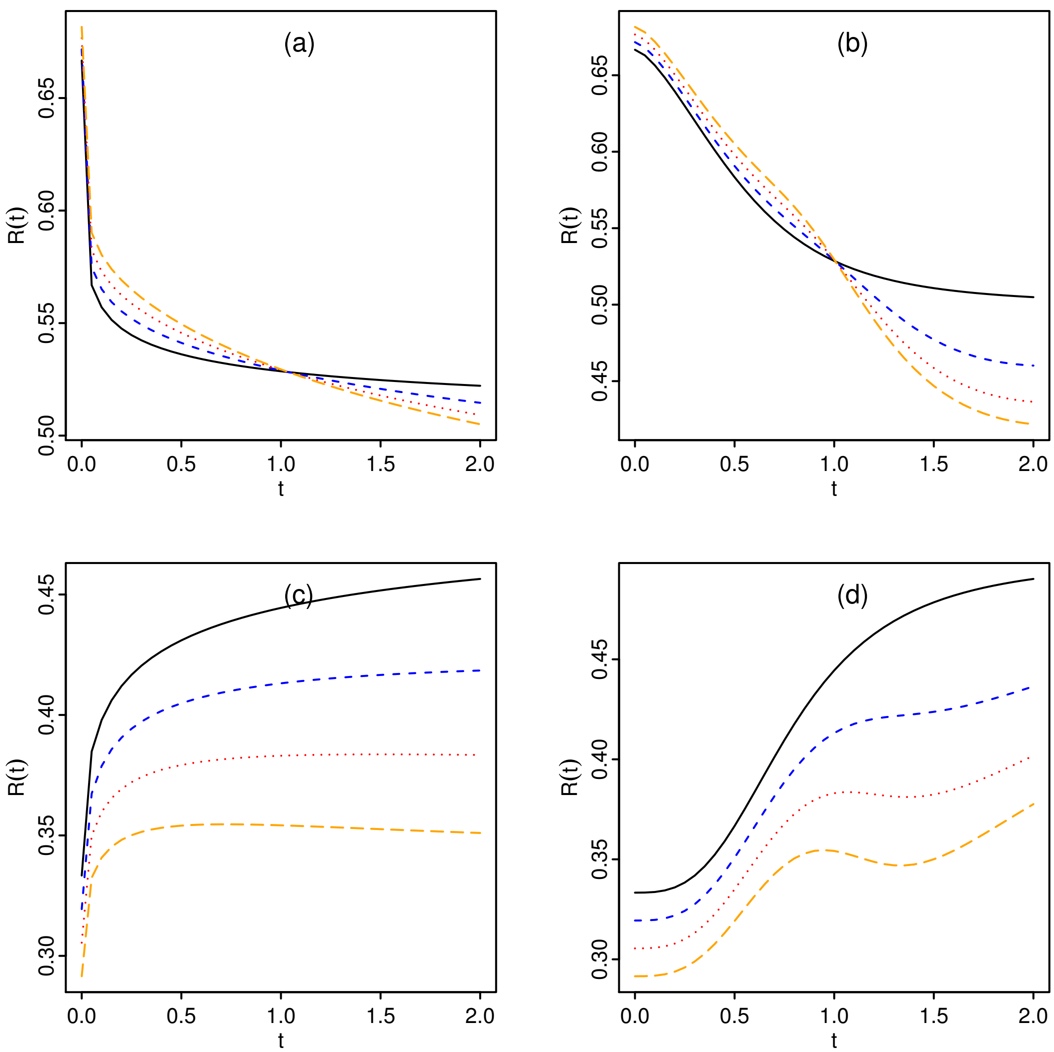

Figure 5: Plot of $R(t)$ based on the GFGM copula with $(m_1, m_2, p_1, p_2) = (5, 5, 2, 1)$, and the Burr III marginal distributions with: (a) $(\alpha, \beta, \delta) = (1, 0.5, 0.5)$, (b) $(\alpha, \beta, \delta) = (1, 0.5, 3)$, (c) $(\alpha, \beta, \delta) = (1, 2, 0.5)$, and (d) $(\alpha, \beta, \delta) = (1, 2, 3)$. Black/solid, blue/dashed, red/dotted, and orange/long-dashed curves relate to $\theta = 0, 0.067, 0.133, 0.200$, respectively.

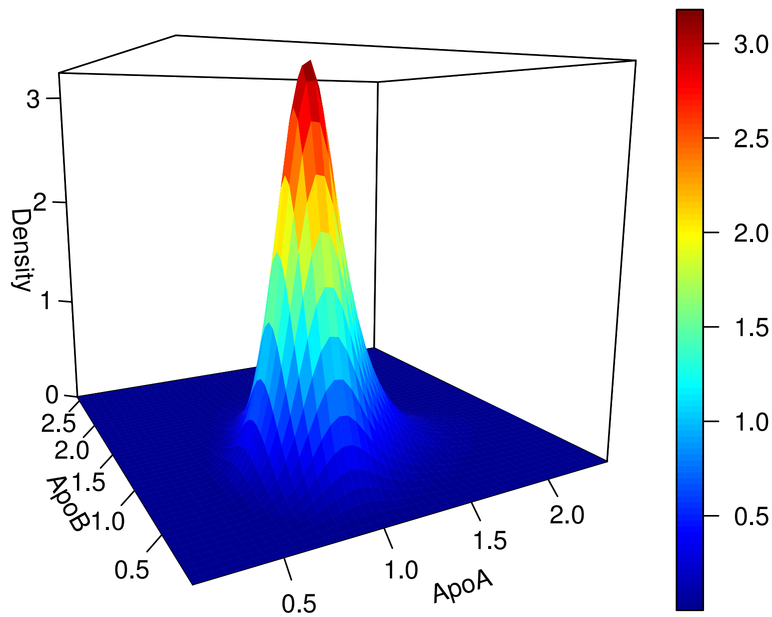

Figure 6: Plot of the PDF constructed using the GFGM copula with $(m_1, m_2, p_1, p_2) = (3, 3, 2, 2)$, and the Burr III marginal distributions with $(\alpha, \beta, \delta) = (1.286, 0.522, 7.398)$.

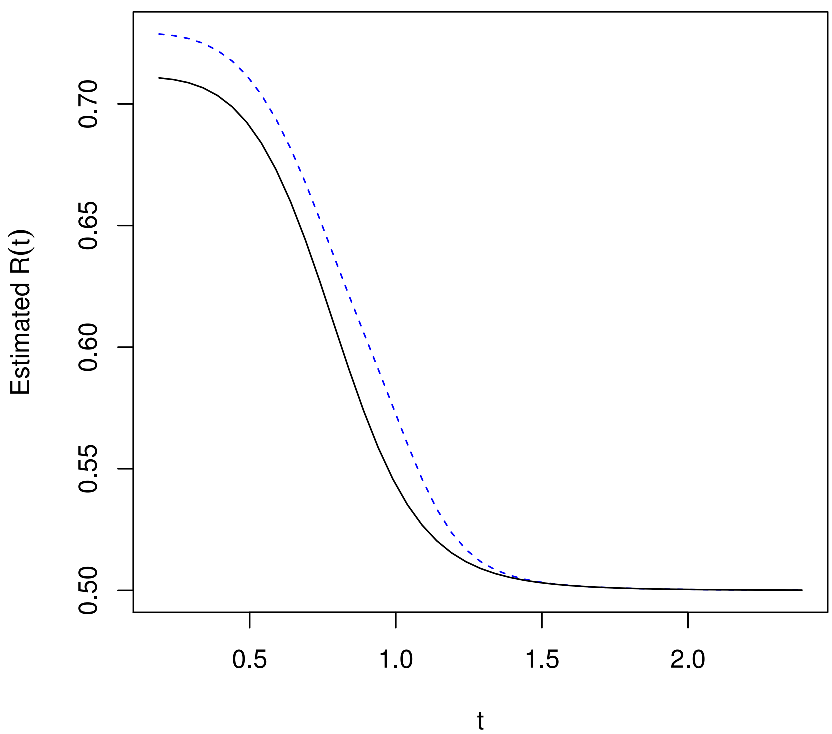

Figure 7: Plot of $R(t)$ estimated from the CHNS data set based on the GFGM copula with $(m_1, m_2, p_1, p_2) = (3, 3, 2, 2)$, and the Burr III marginal distributions with $(\alpha, \beta, \delta) = (1.286, 0.522, 7.398)$. Black/solid and blue/dashed curves relate to $\theta = 0$ and $\theta = 0.176$, respectively.