JACKKNIFE EMPIRICAL LIKELIHOOD INFERENCE FOR THE VARIANCE RESIDUAL LIFE FUNCTION

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Abstract:

- In life testing situations, the residual life time of a component which has survived t units of time is $X_t = X - t|X > t$. In this paper, we give a central limit theorem result for the estimator of $Var(X_t)$, the variance residual life(VRL) function. The result is used to construct normal approximation based confidence interval for the VRL. Furthermore, we use the jackknife empirical likelihood ratio procedure to obtain confidence interval for the VRL function. These intervals are compared through simulation study in terms of the average length and coverage probability. Finally, a numerical example illustrating the theory is also given.

Key-Words:

- *Confidence interval; Coverage probability; Jackknife Empirical likelihood; U-statistic.*

AMS Subject Classification:

- 62N02, 62N05.

1. INTRODUCTION

Let X be a lifetime random variable with distribution function F and survival function $\bar{F} = 1-F$ such that $E(X) < \infty$. The residual life random variable at age t, denoted by $X_t = X - t|X > t$, is simply the remaining lifetime beyond that age. The mean residual life (MRL, also known as the mean remaining life) function is defined formally as $\mu(t) = E(X - t|X > t)$. In industrial reliability studies of repair and replacement strategies, the MRL function may prove to be more relevant than the failure (hazard) rate function. The former summarizes the entire residual life distribution, whereas the latter relates only to the risk of immediate failure. In studies of human populations, demographers often refer the MRL under the names of life expectancy or expectation of life. Obviously, the MRL is of vital importance to actuarial work relating to life insurance policies. For a comprehensive literature review about the MRL see Lai and Xie [21].

Another function which has also generated some interest in the recent years is the variance residual life function defined as $\sigma^2(t) = Var(X - t|X > t)$, see for example, Launer [23] and Gupta et al. [11]. An alternative expression for the residual variance in above is given by

$$\sigma^2(t) = E[(X_t-\mu(t))^2] = \frac{1}{\bar{F}(t)}\int_t^\infty (x-t-\mu(t))^2 dF(x) = \frac{2}{\bar{F}(t)}\int_t^\infty \bar{F}(x)\mu(x)dx-\mu^2(t),$$

where $\mu(t)$ is the mean residual life function.

Numerous research works reveal the importance of the VRL function as a reliability function useful in inference procedures and characterizations, and as a means to classify lifetime distribution using its mathematical behaviour. $\sigma^2(t)$ appears in the formula for $Var(\hat{\mu}_n(t))$, where $\hat{\mu}_n(t)$ is an estimator of the MRL function, see Hall and Wellner [15]. It also appears in the expression of weights assigned for censored observations, see Schmee and Hahn [29]. Launer [23] used $\sigma^2(t)$ to define certain new classes of life distributions and to provide bounds for the reliability function for certain specified class of distributions. Gupta et al. [11] shew that the bihaviour of the VRL function is intimately connected to the behaviour of the mean residual life function of the equilibrium distribution. Lynn and Singpurwalla [25] viewed the burn-in concept as a process of reduction of uncertainty of the lifetime of a component. One approach to this is to minimize the VRL. Combining this with maximizing the MRL leads Block et al. [5] to consider balancing mean and variance residual life through minimizing the residual coefficient of variation (CV). Characterizations of distributions using the VRL function can be found in Huang and Su [16] and references therein.

The role and properties of the variance residual life and the residual coefficient of variation in reliability have been discussed considerably for continuous lifetime random variables by various authors such as Gupta and Kirmani [12], [13], [14], El-Arishi [8], Al-Zahrani and Stoyanov [4] and Abu-Youssef [1], [2], [3]. Gupta [9], [10] studied the VRL, its monotonicity and the associated aging classes of lifetime distributions. Karlin [19] has studied the monotonic behaviour of $\sigma^2(t)$ when the density is log-convex(log-concave). Kanwar and Madhu [18] gave a test for the VRL. Khorashadizadeh, et al. [20] studied properties of the VRL in discrete case. Some stochastic orders have also been defined based on the VRL function (cf. Lai and Xie, [21], p. 61).

Empirical Likelihood (EL) method was originally introduced by Thomas and Grunkkemeier [31] and Owen [26] as a method for constructing nonparametric confidence intervals. During the past decades, the EL method has developed as a very competitive nonparametric test procedure for quite general settings, including the test of a parameter defined by $\int g(t)dF(t)$ with censored survival data (see, e.g., Owen, [27]; Zhao and Qin, [33]; Zhou and Jeong, [34] and the references therein). Inference based on EL has many attractive properties: typically, it does not require estimation of any variance, the range of the parameter space is automatically respected, confidence regions have greater accuracy than those based on the normal approximation approach, furthermore, it inherits all the good properties of the likelihood ratio test and can handle more general types of censored data.

Empirical likelihood has been widely utilized in many settings. However, there exist a lot of computational difficulties when applied to complicated nonlinear functional. To overcome the computational difficulties, a modified EL method was proposed by Jing et al. [17], which was called jackknife empirical likelihood (JEL). The main idea of the JEL is to turn the statistic of interest into a sample mean based on jackknife pseudo-values (see Quenouille, [28]). The goal of this paper is to develop the jackknife empirical likelihood (JEL) method for interval estimation of the VRL function.

The rest of the paper is organized as follows. A U-statistic based estimator of the VRL, the asymptotic normality of the estimator and the corresponding confidence interval/band are given in Section 2. In this Section, we also propose a jackknife empirical likelihood, an adjusted jackknife empirical likelihood for the VRL function, finding better interval estimators of the VRL function. In Section 3, performance of the jackknife empirical likelihood ratio confidence intervals is compared with the normal approximation based ones in terms of coverage probability and average length through a simulation study. Section 4 looks at a real data example illustrating the methods and finally, some concluding remarks are given in Section 5.

2. INFERENCE METHODS

In this section we give the normal approximation based interval for the VRL function. We also develop new interval estimator using jackknife EL methods. In order to overcome the potential undercoverage problem that the JEL methods may encounter as observed in Jing et al. [17], we further propose the adjusted jackknife empirical likelihood by adding one more pseudo-value.

2.1. Normal approximation method

First, note that $\sigma^2(t)$ can be rewritten as

$$\sigma^2(t) = \frac{1}{\bar{F}^2(t)}[\bar{F}(t)\int_t^\infty x^2 dF(x) - (\int_t^\infty x dF(x))^2].$$

Then, given a random sample $X_1, \ldots, X_n$ from the population of X with distribution function F, the VRL function can be estimated as a ratio of two U-statistics

$$U_n^{(1)} = \frac{1}{\binom{n}{2}} \sum_{1 \le i < j \le n} \phi_t^{(1)}(X_i, X_j)$$

and

$$U_n^{(2)} = \frac{1}{\binom{n}{2}} \sum_{1 \le i < j \le n} \phi_t^{(2)}(X_i, X_j)$$

with the symmetric kernels $\phi_t^{(1)}(X_1, X_2) = [0.5(X_1^2 + X_2^2) - X_1 X_2] I(X_1 > t) I(X_2 > t)$ and $\phi_t^{(2)}(X_1, X_2) = I(X_1 > t) I(X_2 > t)$, that is

$$\hat{\sigma}_n^2(t) = \frac{U_n^{(1)}}{U_n^{(2)}},$$

where, $I(.)$ is the indicator function. The following theorem gives the asymptotic distribution of $\hat{\sigma}_n^2(t)$.

Theorem 2.1. *Assume that $E(X^4) < \infty$. Then*

$$\sqrt{n}(\hat{\sigma}_n^2(t) - \sigma^2(t)) \xrightarrow{d} N(0, \upsilon^2(t)),$$

($\xrightarrow{d}$ represents convergence in distribution). $N(0, \upsilon^2(t))$ represents the normal random variable with mean 0 and variance

$$\upsilon^2(t) = 4[\frac{\mu_4(t)}{4\bar{F}^2(t)} + \frac{2\mu_1^2(t)\mu_2(t)}{\bar{F}^4(t)} - \frac{\mu_1^4(t)}{\bar{F}^5(t)} - \frac{\mu_1(t)\mu_3(t)}{\bar{F}^3(t)} - \frac{\mu_2^2(t)}{4\bar{F}^3(t)}],$$

where $\mu_i(t) = \int_t^\infty x^i dF(x)$, $i = 1, 2, 3, 4$.

Proof: The result immediately follows from Theorem 6.1.6 in Lehmann ([24], p. 376) and the standard delta method. □

It is obvious that $\upsilon^2(t)$ can be consistently estimated by its empirical counterpart,

$$\hat{\upsilon}_n^2(t) = 4[\frac{\hat{\mu}_4(t)}{4\bar{F}_n^2(t)} + \frac{2\hat{\mu}_1^2(t)\hat{\mu}_2(t)}{\bar{F}_n^4(t)} - \frac{\hat{\mu}_1^4(t)}{\bar{F}_n^5(t)} - \frac{\hat{\mu}_1(t)\hat{\mu}_3(t)}{\bar{F}_n^3(t)} - \frac{\hat{\mu}_2^2(t)}{4\bar{F}_n^3(t)}] I(X_{(n)} > t),$$

where $F_n(t) = \frac{1}{n}\sum_{i=1}^n I(X_i \leq t)$ is the empirical distribution function, $\bar{F}_n = 1 - F_n$,

$$\hat{\mu}_i(t) = \int_t^\infty x^i dF_n(x) = \frac{1}{n}\sum_{j=1}^n X_j^i I(X_j > t), \quad i = 1, 2, 3, 4,$$

and $X_{(n)} = \max\{X_1, \ldots, X_n\}$. Thus, an asymptotic $100(1-\alpha)\%$ confidence interval for $\sigma^2(t)$ at fixed time t based on the above normal approximation can be given by

$$\{\sigma^2(t) : n(\hat{\sigma}_n^2(t) - \sigma^2(t))^2 \leq \hat{v}^2(t)\chi^2_{1-\alpha}(1)\},$$

where $\chi^2_{1-\alpha}(1)$ is the $100(1-\alpha)$-percentile of the chi-square distribution with one degree of freedom.

The following theorem gives the weak convergence of the stochastic process based on $\hat{\sigma}_n^2(t)$ which can be used to construct a simultaneous confidence band for $\sigma^2(t)$. Let $b < \infty$ and $b \in [0, \tau]$, where $\tau = \inf\{t : F(t) = 1\}$ and denote

$$\rho(s,t) = E[(X - s - \mu(s))^2(X - t - \mu(t))^2 I(X > t)],$$

$$\nu(s,t) = \int_t^\infty (x - s - \mu(s))^2 dF(x).$$

Theorem 2.2. Suppose that $E(X^4) < \infty$. Then the process $\sqrt{n}(\hat{\sigma}_n^2(t) - \sigma^2(t))$ for $t \in [0, b]$ converges in distribution to a Gaussian process $U(t)$ with mean zero and covariance function

$$\begin{aligned}\Gamma(s,t) &= \frac{1}{\bar{F}(s)\bar{F}(t)}[\rho(b,b) - \rho(t,b) - \rho(s,b) + \rho(s,t) - \bar{F}^4(b)\sigma^4(b) \\ &+ \bar{F}(s)\bar{F}(b)\sigma^2(s)\sigma^2(b) + \bar{F}(t)\bar{F}(b)\sigma^2(t)\sigma^2(b) - \sigma^2(t)\nu(s,t)],\end{aligned}$$

where $0 \leq s \leq t \leq b$.

Proof: First note that the estimator $\hat{\sigma}_n^2(t)$ can also be given by

$$\begin{aligned}\hat{\sigma}_n^2(t) &= \frac{1}{n\bar{F}_n(t)}\sum_{i=1}^n (X_i - t - \mu_n(t))^2 I(X_i > t) \\ &= \frac{1}{n\bar{F}_n(t)}\sum_{i=1}^n (X_i - t - \mu(t))^2 I(X_i > t) - [\mu_n(t) - \mu(t)]^2,\end{aligned}$$

where $\mu_n(t) = \frac{1}{\bar{F}_n(t)}\int_t^\infty \bar{F}_n(x)dx$ is the empirical estimator of the mean residual life function. Then

$$\sqrt{n}(\hat{\sigma}_n^2(t) - \sigma^2(t)) = \frac{1}{\bar{F}_n(t)}\{V_n(t) - \sigma^2(t)\sqrt{n}[\bar{F}_n(t) - \bar{F}(t)]\} - \sqrt{n}[\mu_n(t) - \mu(t)]^2,$$

where

$$V_n(t) = n^{-\frac{1}{2}}\sum_{i=1}^n [(X_i - t - \mu_n(t))^2 I(X_i > t) - \sigma^2(t)\bar{F}(t)].$$

Applying the same procedure of proof of Lemma 3 in Yang [32] follows that $V_n(t)$ weakly converges to a Gaussian process $V(t)$ with $E[V(t)] = 0$ and

$$\begin{aligned} E[V(s)V(t)] &= \rho(b,b) - \rho(t,b) - \rho(s,b) + \rho(s,t) - \bar{F}^4(b)\sigma^4(b) \\ &+ \bar{F}(s)\bar{F}(b)\sigma^2(s)\sigma^2(b) + \bar{F}(t)\bar{F}(b)\sigma^2(t)\sigma^2(b) - \sigma^2(s)\sigma^2(t)\bar{F}(s)\bar{F}(t), \end{aligned}$$

where $0 \le s \le t \le b$. On the other hand, Theorem 1 in Yang [32] implies that $\sqrt{n}[\mu_n(t) - \mu(t)]^2 = o_p(1)$, uniformly in $t \in [0, b]$. The result now follows from the fact that $\sqrt{n}[\bar{F}_n(t) - \bar{F}(t)]$ converges to a Brownian bridge and $\bar{F}_n^{-1}(t) \to \bar{F}^{-1}(t)$ uniformly in $t \in [0, b]$ with probability one. □

Theorem 2.2 can be used to obtain the following confidence band for $\sigma^2(t)$. By the continuous mapping theorem we have

$$\sup_{0\le t\le b} \{\sqrt{n}(\hat{\sigma}_n^2(t) - \sigma^2(t))\} \xrightarrow{d} \sup_{0\le t\le b} U(t).$$

Now, we can define the asymptotic $100(1-\alpha)\%$ simultaneous confidence band for $\sigma^2(t)$ in $t \in [0, b]$ as follows,

$$\{\sigma^2(t) : \sqrt{n}(\hat{\sigma}_n^2(t) - \sigma^2(t)) \le c_\alpha\},$$

where c_α is the upper α-percentile of the distribution of $\sup_{0\le t\le b} U(t)$.

2.2. Jackknife empirical likelihood method

In this subsection, we construct a confidence interval for the true $\sigma^2(t)$ via jackknife empirical likelihood (JEL). Let $X_1, \ldots, X_n (n \ge 2)$ be a random sample from a distribution function F. We define a one-sample U-statistic of degree 2

$$U_n(\sigma^2(t)) = \frac{1}{\binom{n}{2}} \sum_{1\le i<j\le n} \phi_t(X_i, X_j; \sigma^2(t)),$$

with symmetric kernel

$$\phi_t(X_1, X_2; \sigma^2(t)) = [\sigma^2(t) + X_1X_2 - 0.5(X_1^2 + X_2^2)]I(X_1 > t)I(X_2 > t).$$

It is easy to check that $E[U_n(\sigma^2(t))] = 0$, for the true $\sigma^2(t)$. To apply the JEL, we define our jackknife pseudo-values by

$$\hat{V}_i(\sigma^2(t)) = nU_n(\sigma^2(t)) - (n-1)U_{n-1}^{(-i)}(\sigma^2(t)),$$

where $U_{n-1}^{(-i)}$ is the U-statistic after deleting the ith observation X_i. It can be easily shown that $E[\hat{V}_i] = 0$ and

$$U_n(\sigma^2(t)) = \frac{1}{n}\sum_{i=1}^{n} \hat{V}_i(\sigma^2(t)).$$

Then, one can apply the standard EL method to $\hat{V}_i$. Let $\mathbf{p} = (p_1, \ldots, p_n)$ be the probability vector over $\hat{V}_i$. The jackknife empirical likelihood ratio at true value $\sigma^2(t)$ is defined by

$$R(\sigma^2(t)) = \max\{\prod_{i=1}^n np_i : p_i \geq 0, i = 1, \ldots, n, \sum_{i=1}^n p_i = 1, \sum_{i=1}^n p_i \hat{V}_i(\sigma^2(t)) = 0\}.$$

By using the standard Lagrange multiplier method, we know that $R(\sigma^2(t))$ is maximized when

$$p_i = \frac{1}{n}\{1 + \lambda \hat{V}_i(\sigma^2(t))\}^{-1}, i = 1, \ldots, n,$$

where $\lambda = \lambda(\sigma^2(t))$ satisfies

$$\frac{1}{n}\sum_{i=1}^n \frac{\hat{V}_i(\sigma^2(t))}{1 + \lambda \hat{V}_i(\sigma^2(t))} = 0.$$

Let $g(x) = E[\phi_t(x, X_2; \sigma^2(t))]$ and $\sigma_g^2 = Var(g(X_1))$. Now we have Wilks theorem for the JEL as follows.

Theorem 2.3. *Assume that $E(X^4) < \infty$ and $\sigma_g^2 > 0$. Then, as $n \to \infty$*

$$-2\log R(\sigma^2(t)) \xrightarrow{d} \chi_1^2,$$

where χ_1^2 is a chi-distribution with one degree of freedom.

Theorem 2.3 is a special case of Theorem 1 in Jing et al. [17] with $m = 2$. Instead of the regularity condition $E[\phi_t^2(X_1, X_2; \sigma^2(t))]$ required by Theorem 1 in Jing et al. [17], Theorem 2.3 requires existence of the forth moment because of the specific form of the VRL function.

Following this, an asymptotic $100(1-\alpha)\%$ confidence interval for $\sigma^2(t)$ at time t can be given by

$$\{\tilde{\sigma}^2(t) : -2\log R(\tilde{\sigma}^2(t)) \leq \chi_{1-\alpha}^2(1)\},$$

where $\chi_{1-\alpha}^2(1)$ is the is $100(1-\alpha)$-percentile of the chi-square distribution with one degree of freedom.

From practical point of view, the function `el.cen.EM2` inside the package `emplik`, which is an extension package to be used with the R software, carries out calculating the above confidence interval.

Remark 2.1. Using the same procedure as the proof of Theorem 2.2 of Zhao and Qin [33] and following Theorem 2.1 of Jing et al. [17], the above Theorem 2.2 implies that

$$-2\log R(\sigma^2(t)) \xrightarrow{d} \frac{W(t)}{4\sigma_g^2},$$

where $W(t)$ is a Gaussian process with mean zero and covariance function

$$Cov(W(s), W(t)) = \bar{F}(s)\bar{F}(t)\Gamma(s,t).$$

Thus, an JEL-based asymptotic $100(1-\alpha)\%$ simultaneous confidence band for $\sigma^2(t)$ in $t \in [0, b]$ can be given by

$$\{\tilde{\sigma}^2(t) : -2\log R(\tilde{\sigma}^2(t)) \leq k_\alpha\},$$

where k_α is the upper α-percentile of the distribution of $\sup_{0\leq t\leq b} \frac{W(t)}{4\sigma_g^2}$.

2.3. Adjusted Jackknife empirical likelihood method

Chen et al. [7] developed an adjusted empirical likelihood method, which significantly improves the performance of the empirical likelihood method in terms of coverage probability when the sample size is not large. We adapt their approach to the JEL for $\sigma^2(t)$ by adding one more jackknife pseudo-value

$$\hat{V}_{n+1}(\sigma^2(t)) = -\frac{a_n}{n}\sum_{i=1}^{n} \hat{V}_i(\sigma^2(t)),$$

for constant $a_n = \max\{1, \frac{1}{2}\log(n)\}$. The adjusted jackknife empirical likelihood (AJEL) ratio of $\sigma^2(t)$ is given by

$$R^{ad}(\sigma^2(t)) = \max\{\prod_{i=1}^{n+1}(n+1)p_i : p_i \geq 0, i = 1, \ldots, n+1, \sum_{i=1}^{n+1} p_i = 1, \sum_{i=1}^{n+1} p_i\hat{V}_i(\sigma^2(t)) = 0\}.$$

With the same conditions given by Jing et al. [17], Wilks theorem of the AJEL has been established by Chen and Ning [6]. Thus, as a special case, the following theorem holds for the above AJEL ratio. For the proof, we refer the reader to Chen and Ning [6].

Theorem 2.4. *Assume that $E(X^4) < \infty$ and $\sigma_g^2 > 0$. Then, as $n \to \infty$*

$$-2\log R^{ad}(\sigma^2(t)) \xrightarrow{d} \chi_1^2.$$

A $100(1-\alpha)\%$ confidence interval for $\sigma^2(t)$ by the adjusted JEL method can be developed similarly.

3. SIMULATION STUDY

Simulation exercises were undertaken to assess the performance of the normal approximation (NA) based confidence interval, comparing with the jackknife empirical likelihood (JEL) and adjusted jackknife empirical likelihood (AJEL) confidence intervals in terms of the average length and coverage probability. In the simulation, we considered the following two models for the underling lifetime distribution of X:

(i) X is uniformly distributed on $(0, 1)$,

(ii) X has a Weibull distribution with survival function $\bar{F}(x) = e^{-\frac{1}{2}x^2}$.

One can readily show that in case (i)

$$\sigma^2(t) = \frac{1}{3(1-t)}(1 - 3t + 3t^2 - t^3) - \frac{1}{4}(1-t)^2,$$

and in case (ii)

$$\sigma^2(t) = 2[1 - \frac{t\bar{\Phi}(t)}{\phi(t)}] - 2\pi e^{t^2}\bar{\Phi}^2(t),$$

where $\phi(t)$ and $\bar{\Phi}(t)$ refer to the standard normal density and survival function, respectively. In each case, we ran 2000 simulation trials of different sample sizes $n = 50, 100$ and 150 to obtain confidence intervals with nominal confidence level of 0.95. We compute the average length of intervals and coverage probabilities, i.e. the proportion of intervals which cover the true value $\sigma^2(t)$ for different values of t.

Table 1 - Table 2 summarize the results of the 2000 simulation trials for both models. From the tables, as the sample size n increases, all methods improve in terms of coverage probabilities. It is also evident from the tables that, specially in Weibull model, the coverage probability of the NA confidence interval is not satisfied when the sample size is small and moderate. However, JEL and AJEL produce slightly better coverage probabilities for the same sample size. When the sample size is large, NA, JEL and AJEL methods have similar performance in terms of coverage probability. We can see coverage probability for AJEL is very close to nominal level 0.95, and AJEL has better performance than JEL for the small sample size. Though, for large values of t, the coverage probability of all the methods is slightly far from the nominal level.

For all the methods, the length of confidence interval becomes shorter when the sample size becomes larger. When the sample size increases from moderate to large, the length of confidence interval for all the methods are very close. It seems that, for large values of t, the length of the NA confidence intervals is slightly shorter than JEL and AJEL confidence intervals.

n	Method	$t=0$	$t=0.2$	$t=0.4$	$t=0.6$	$t=0.8$
	NA	0.930	0.935	0.923	0.906	0.822
		(0.041)	(0.029)	(0.019)	(0.010)	(0.003)
50	JEL	0.929	0.935	0.924	0.888	0.908
		(0.040)	(0.029)	(0.018)	(0.010)	(0.068)
	AJEL	0.940	0.946	0.937	0.908	0.946
		(0.042)	(0.030)	(0.019)	(0.011)	(0.089)
	NA	0.939	0.936	0.930	0.928	0.900
		(0.029)	(0.021)	(0.013)	(0.007)	(0.002)
100	JEL	0.935	0.930	0.930	0.922	0.919
		(0.028)	(0.020)	(0.013)	(0.007)	(0.003)
	AJEL	0.941	0.938	0.935	0.931	0.932
		(0.029)	(0.021)	(0.013)	(0.007)	(0.003)
	NA	0.941	0.936	0.944	0.938	0.920
		(0.024)	(0.017)	(0.011)	(0.006)	(0.002)
150	JEL	0.937	0.932	0.943	0.940	0.947
		(0.022)	(0.016)	(0.011)	(0.006)	(0.002)
	AJEL	0.941	0.937	0.947	0.942	0.950
		(0.022)	(0.017)	(0.011)	(0.006)	(0.002)

Table 1: Empirical coverage probabilities (average length) for $\sigma^2(t)$, uniform model

n	Method	$t=0$	$t=0.25$	$t=0.5$	$t=1$	$t=1.7$
	NA	0.887	0.878	0.864	0.834	0.679
		(0.328)	(0.316)	(0.300)	(0.277)	(0.250)
50	JEL	0.909	0.903	0.894	0.862	0.870
		(0.304)	(0.306)	(0.302)	(0.296)	(0.426)
	AJEL	0.924	0.910	0.908	0.873	0.896
		(0.315)	(0.318)	(0.314)	(0.310)	(0.480)
	NA	0.913	0.924	0.908	0.886	0.778
		(0.239)	(0.236)	(0.223)	(0.214)	(0.207)
100	JEL	0.925	0.929	0.919	0.917	0.834
		(0.240)	(0.240)	(0.167)	(0.154)	(0.241)
	AJEL	0.935	0.935	0.928	0.923	0.845
		(0.245)	(0.246)	(0.170)	(0.157)	(0.251)
	NA	0.927	0.926	0.923	0.909	0.811
		(0.200)	(0.193)	(0.185)	(0.179)	(0.187)
150	JEL	0.929	0.938	0.929	0.926	0.859
		(0.200)	(0.144)	(0.132)	(0.184)	(0.201)
	AJEL	0.934	0.940	0.934	0.930	0.865
		(0.203)	(0.146)	(0.134)	(0.187)	(0.205)

Table 2: Empirical coverage probabilities (average length) for $\sigma^2(t)$, Weibull model

4. REAL DATA ANALYSIS

In this section, we use a real data coming from reliability engineering to illustrate applications of the NA-based and JEL-based confidence intervals for the VRL function. Since the variance estimator $\hat{\upsilon}^2(t)$ is unstable, the NA-based confidence interval for the VRL contains negative values. In the following computation results, the values outside of the positive range of the VRL are removed and the negative lower bounds of the confidence intervals are replaced with zero.

Lawless [22] used the breaking strengths of single carbon fibers of different to fit a parametric regression model. We use the data set consisting of breaking strengths of 57 single carbon fibers with unit length taken from Lawless [22] to estimate $\sigma^2(t)$. Table 3 gives the estimated VRL function and corresponding 95% lower bound (LB), upper bound (UB) and length based on the NA, JEL and AJEL methods at different time points t. We can see from the table that the lengths of confidence intervals for the NA is longer than one for the JEL and AJEL methods. Also, there is no big difference among the lengths of the JEL and AJEL confidence intervals.

		NA			JEL			AJEL		
t	VRL	LB	UB	Length	LB	UB	Length	LB	UB	Length
0.5	0.697	0	2.503	3.612	0.477	0.998	0.521	0.472	1.015	0.543
2.5	0.637	0	2.232	3.189	0.443	0.901	0.458	0.433	0.911	0.477
3.5	0.432	0	1.627	2.389	0.291	0.643	0.352	0.284	0.654	0.370
4.5	0.250	0	1.057	1.615	0.145	0.413	0.268	0.138	0.423	0.285
5.0	0.130	0	0.620	0.979	0.001	1.129	1.128	0.328	1.129	0.801

Table 3: Estimated variance residual lifetimes, 95% confidence intervals and lengthes, carbon fiber data

5. CONCLUSION

In this paper, we have considered an estimator of the VRL function. The estimator was shown to converge in distribution to a normal random variable. Furthermore, a confidence interval for the VRL function at time t was constructed by using the normal approximation (NA) method. As alternative methods, we have also considered constructing confidence interval/band for the VRL function using the jackknife empirical likelihood (JEL) and adjusted jackknife empirical likelihood (AJEL) approaches. A major advantage of the EL-based method is no need for nonparametric estimation of any kind of variance for statistical inference. A simulation exercise was undertaken to compare between the performance of the

NA-based and El-based confidence intervals in terms of coverage probabilities and the average lengthes. As shown from the simulation study, the coverage probability for the NA method is far away from our expectation when the sample size is small. However, the coverage probability of confidence intervals for JEL and AJEL methods is very close to nominal level. The length of confidence interval for all the methods is very close when the sample size increases from moderate to large. Finally, using a numerical example, the application of the methods for constructing confidence intervals was illustrated.

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REFERENCES

[1] Abu-Youssef, S. E. (2004). Nonparametric test for monotone variance residual life class of life distributions with hypothesis testing applications, *Applied Mathematics and Computations,* **158**, 817-826.

[2] Abu-Youssef, S. E. (2007). Testing decreasing (increasing) variance residual class of life distributions using kernel method, *Applied Mathematical Sciences,* **1**, 1915-1927.

[3] Abu-Youssef, S. E. (2009). A Goodness of fit approach to monotone variance residual life class of life distributions, *Applied Mathematical Sciences,* **3**, 15, 715-724.

[4] Al-Zahrani, B. and Stoyanov, J. (2008). On some properties of life distributions with increasing elasticity and log-concavity, *Applied Mathematical Sciences,* **2**, 48, 2349-2361.

[5] Block, H. W.; Savits, T. H. and Singh, H. (2002). A Criterion for Burn-in That Balances Mean Residual Life and Residual Variance, *Operations Research,* **50**, 290-296.

[6] Chen, Y. J. and Ning, W. (2016). Adjusted jackknife empirical likelihood, *arXiv:1603.04093v1.*

[7] Chen, J.; Variyath, A. M. and Abraham, B. (2008). Adjusted empirical likelihood and its properties, *Journal of Computational and Graphical Statistics,* **17**, 426-443.

[8] El-Arishi, S. (2005). A conditional variance characterization of some discrete probability distributions, *Statistical Papers,* **46**, 31-45.

[9] Gupta, R. C. (1987). On the monotonic properties of the residual variance and their application in reliability, *Journal of Statistical Planning and Inference,* **16**, 329-335.

[10] Gupta, R. C. (2006). Variance residual life function in reliability studies, *Metron-International Journal of Statistics*, **LXIV**, 3, 343–355.

[11] Gupta, R. C.; Kirmani, S. N. U. A. and Launer, R. L. (1987). On life distributions having monotone residual variance, *Probability in the Engineering and Informational Sciences,* **1**, 299-307.

[12] Gupta, R. C. and Kirmani, S. N. U. A. (1998). On the proportional mean residual life model and its implications, *Statistics,* **32**, 175-187.

[13] Gupta, R. C. and Kirmani, S. N. U. A. (2000). Residual coefficient of variation and some characterization results, *Journal of Planning and Statistical Inference,* **91**, 23-31.

[14] Gupta, R. C. and Kirmani, S. N. U. A. (2004). Moments of residual life and some characterization results, *Journal of Applied Statistical Science,* **13**, 2, 155-167.

[15] Hall, W. J. and Wellner, J. A. (1981). *Mean residual life.* In "Proceedings of the International Symposium on Statistics and Related Topics" (M. Csorgo, D. A. Dawson, J. N. K. Rao and A. K. Md. E. Saleh, Eds.), North-Holland Publishing Co., Amsterdam, 169-184.

[16] Huang, W. J. and SU, N. C. (2012). Characterizations of Distributions Based on Moments of Residual Life, *Communications in StatisticsTheory and Methods,* **41**, 15, 2750–2761.

[17] Jing, B.; Yuan, Q. and Zhou, W. (2009). Jackknife empirical likelihood, *Journal of the American Statistical Association,* **104**, 1224-1232.

[18] Kanwar, S. and Madhu, B. J. (1991). A test for the variance residual life, *Communications in StatisticsTheory and Methods,* **20**, 1, 327–331.

[19] Karlin, S. (1982). *Some results on optimal partinioning of variance and monotonicity with truncation level.* In "Statistics and Probability: Essays in honor of C. R. Rao" (G. kallianpur, p. R. Krishnaiah, and J. K. Ghosh, Eds.), North Holland Publishing Co., Amsterdam, 375-382.

[20] Khorashadizadeh, M.; Rezaei Roknabadi, A. H. and Mohtashami Borzadran, G. R. (2010). Variance residual life function in discrete random ageing, *Metron-International Journal of Statistics*, **LXVIII**, 1, 67–75.

[21] Lai, C. and Xie, M. (2006). *Stochastic Ageing and Dependence for Reliability,* Springer, New York.

[22] Lawless, J. (2003). *Statistical Models and Methods for Lifetime Data,* John Wiley and Sons, New Jersey.

[23] Launer, R. L. (1984). Inequalities for NBUE and NWUE life distributions, *Operations Research,* **32**, 660-667.

[24] Lehmann, E. L. (1999). *Elements of Large-Sample Theory,* Springer-Verlag, New York.

[25] Lynn, N. J. and Singpurwalla, N. D. (1997). Comment: Burn-in makes us feel good, *Statistical Science,* **12**, 13-19.

[26] Owen A. (1988). Empirical likelihood ratio confidence intervals for a single functional, *Biometrika,* **75**, 237-249.

[27] Owen, A. (2001). *Empirical Likelihood,* Chapman and Hall, London.

[28] Quenouille, M. (1956). Notes on bias in estimation, *Biometrika,* **10**, 353-360.

[29] Schmee, J. and Hahn, G. J. (1979). A Simple Method for Regression Analysis with Censored Data, *Technometrics,* **21**, 4, 417–432.

[30] Sen, K. and Jain, M. B. (1991). A test for the variance residual life, *Communications in Statistics - Theory and Methods,* **20**, 1, 327–331.

[31] Thomas D. R. and Grunkemeier, G. L. (1975). Confidence interval estimation of survival probabilities for censored data, *Journal of the American Statistical Association*, **70**, 865-871.

[32] Yang, G. L. (1978). Estimating of a biometric function, *Annals of Statistics,* **6**, 1, 112-116.

[33] Zhao, Y. and Qin, G. (2006). Inference for the mean residual life function via empirical likelihood, *Communications in Statistics: Theory and Methods,* **35**, 1025-1036.

[34] Zhou, M. and Jeong, J. H. (2011). Empirical likelihood ratio test for median and mean residual lifetime, *Statistics in Medicine,* **30**, 2, 152-159.