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MODELING RISK OF EXTREME EVENTS IN GENERALIZED VERHULST MODELS

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Abstract:

- A very popular model in population dynamics, which has been around since the first half of the nineteenth century, is the Verhulst logistic model. However, some limitations of this model have provided grounds to propose more sophisticated growth models using, for instance, the former as a basis. Since the Verhulst model and some generalizations of it are closely connected to extreme value distributions, either max-geometric-stable or max-stable, we show that the parameter attached to the retroaction factor of these generalized models establishes, on its own, which extreme value distribution is adequate to model risks of extreme events in population dynamics.

Key-Words:

- *Extreme value distributions; generalized Verhulst models; growth and retroaction parameters; max-geometric-stable distributions; population dynamics.*

AMS Subject Classification:

- 60G70, 92D25.

1. INTRODUCTION

> *"It is generally agreed that the specific growth rate [...] declines as density increases, and hence that the form of the population curve with time in a limited system has a sigmoid shape. Of the many proposed models only one, the logistic of Verhulst (1838) [...] is widely used. It is presented in most current ecology texts and is incorporated into almost all fish and game management theories. Such tacit acceptance probably derives from its mathematical simplicity and biological clarity."*
>
> Smith (1963)

Let $N(t)$ be the size of a population and $R(t)$ the amount of available resources at time t. It is reasonable to relate $R(t)$ and $N(t)$ by the differential equation

$$\frac{\mathrm{d}}{\mathrm{d}t}R(t) = -\eta\frac{\mathrm{d}}{\mathrm{d}t}N(t)\,, \tag{1.1}$$

with η representing the amount of resources consumed to yield a new population unit. The solution of (1.1) is

$$R(t) = \eta\big(K - N(t)\big) = R(0) - \eta N(t)\,,$$

and hence $K = R(0)/\eta > 0$ is the carrying capacity, *i.e.* the limiting size the population may reach without disruptive effects on the availability of resources.

On the other hand, it also makes sense to consider that the population growth rate is proportional to the amount of available resources, namely

$$\frac{\frac{\mathrm{d}}{\mathrm{d}t}N(t)}{N(t)} = \mu R(t)\,.$$

Therefore,

$$\frac{\mathrm{d}}{\mathrm{d}t}N(t) = \rho\,N(t)\left(1 - \frac{N(t)}{K}\right)\,, \tag{1.2}$$

where $\rho = \mu R(0) > 0$ is the malthusian intrinsic growth rate, or growth rate *per capita*. In the right side of equation (1.2), $N(t)$ is considered to be the growth factor and $1 - N(t)/K$ the retroaction factor, which is responsible for curbing down population growth to sustainable levels. The solution of (1.2), known as the Verhulst model (Verhulst [16]), is

$$N(t) = \frac{K N(0)}{N(0) + (K - N(0))\,\mathrm{e}^{-\rho t}}\,, \tag{1.3}$$

which belongs to the logistic family of functions, hence the name logistic model ($N(0)$ is the initial population size).

On some occasions, it is more convenient to express the Verhulst logistic equation (1.2) as a function of the population density $\delta(t) = N(t)/K$, namely

$$\frac{\mathrm{d}}{\mathrm{d}t}\delta(t) = \rho\,\delta(t)\big(1-\delta(t)\big)\,. \tag{1.4}$$

The solution of (1.4) is

$$\delta(t) = \frac{1}{1+\exp(-\rho t)}\,,$$

which is a member of the logistic family of distributions. As pointed out in Smith [13], over time the population curve will have a sigmoid shape, which is typical of continuous distribution functions.

In spite of its popularity, the Verhulst model has some limitations. For instance, one limitation is only being suitable for modeling sustainable growth, or modeling stable populations, in the sense that the population sizes are maintained at sustainable levels. Therefore, over the years the Verhulst model has been used as a building block for other more sophisticated models, some of which allowing the possibility of modeling different types of unrestricted population growth.

Many of newer models state that either $\frac{\mathrm{d}}{\mathrm{d}t}N(t)$ or $\frac{\mathrm{d}}{\mathrm{d}t}\ln N(t)$ is a decreasing function of the population density (as does the Verhulst model). As an example, we have the family of models based on the Box-Cox family of transformations (Box and Cox [3])

$$\frac{\mathrm{d}}{\mathrm{d}t}\ln N(t) = \rho\,\frac{1-\left(\frac{N(t)}{K}\right)^{\nu}}{\nu} \;\Leftrightarrow\; \begin{cases} \frac{\mathrm{d}}{\mathrm{d}t}N(t) = \rho\,N(t)\,\frac{1-\left(\frac{N(t)}{K}\right)^{\nu}}{\nu} & ,\nu>0\\ \frac{\mathrm{d}}{\mathrm{d}t}N(t) = \rho\,N(t)\Big(-\ln\left(\frac{N(t)}{K}\right)\Big) & ,\nu=0 \end{cases}, \tag{1.5}$$

which contains the Verhulst model as a special case ($\nu = 1$). The subfamily in (1.5) for $\nu > 0$ was considered in Richards [12], and the solution for $\nu = 0$ is

$$N(t) = K\exp\bigg(\ln\bigg(\frac{N(0)}{K}\bigg)\exp\left(-\rho t\right)\bigg)\,,$$

which is commonly known in population dynamics as the Gompertz growth model. This model is proportional to the Gumbel distribution, a well known extreme value (EV) distribution for maxima, and has been used, for instance, to model the growth of cancer tumors. Note that the Gumbel distribution has the functional form

$$\Lambda(x;\lambda,\delta) = \exp(-\exp(-(x-\lambda)/\delta)),\quad x\in\mathbb{R},\ (\lambda,\delta)\in\mathbb{R}\times\mathbb{R}^+\,, \tag{1.6}$$

where λ and δ are, respectively, location and scale parameters.

A natural extension of Verhulst's equation (1.2) is the Blumberg hiperlogistic equation (Blumberg [2])

$$\frac{\mathrm{d}}{\mathrm{d}t}N(t) = \rho\left(N(t)\right)^{\alpha}\left(1-\frac{N(t)}{K}\right)^{\beta},\quad \alpha,\beta>0\,. \tag{1.7}$$

However, Blumberg's equation does not contain a closed form analytical solution, except for some values of the parameters α and β. For example, if $\alpha+\beta=2$ (and $K=1$), the solution of (1.7) belongs to the class of max-geometric-stable distributions, where the shape parameter is a function of the retroaction parameter β in (1.7).

On the other hand, Brilhante *et al.* [4] extended the subfamily in (1.5) for $\nu=0$ by considering

$$\frac{\mathrm{d}}{\mathrm{d}t}N(t)=\rho\,N(t)\left(-\ln\left(\frac{N(t)}{K}\right)\right)^{1+\xi}\,,\quad \xi\in\mathbb{R}\,. \tag{1.8}$$

Those authors showed that the solution of (1.8), when $K=1$, belongs to the general EV (GEV) family of distributions for maxima, with the functional form

$$G_\xi(x;\lambda,\delta)=\exp\left(-\big(1+\xi(x-\lambda)/\delta\big)^{-1/\xi}\right)\,,\qquad 1+\xi(x-\lambda)/\delta>0, \tag{1.9}$$

where $\xi\in\mathbb{R}$, $\lambda\in\mathbb{R}$ and $\delta\in\mathbb{R}^+$ are, respectively, shape, location and scale parameters. Observe that equation (1.8) can also be considered a generalization of Verhulst's logistic equation, since $1-N(t)/K$ is a linear approximation of $-\ln(N(t)/K)$, due to the fact that $N(t)/K\to 1$, as $t\to\infty$. The effect of replacing $1-N(t)/K$ by $-\ln(N(t)/K)$ in (1.2) is that we shall have a weaker control over population growth than before. This weaker control effect can easily be explained by noticing that if $x\in(0,1)$, $1-x$ is proportional to the density function of the minimum $U_{1:2}=\min(U_1,U_2)$ and $-\ln x$ is the density function of the product U_1U_2, where U_1 and U_2 are two independent standard uniform random variables, and thus the stochastic ordering $U_1U_2\preceq U_{1:2}$ holds true.

An even more general differential equation for population dynamics, based on the BetaBoop family of densities, was considered in Brilhante *et al.* [5], namely

$$\frac{\mathrm{d}}{\mathrm{d}t}N(t)=\rho\,\big(N(t)\big)^{\alpha}\big[-\ln(1-N(t))\big]^{\beta}\big(1-N(t)\big)^{\gamma}\big(-\ln N(t)\big)^{\delta}\,, \tag{1.10}$$

where $\alpha,\beta,\gamma,\delta>0$. The previous equation includes equations (1.7) and (1.8) as special cases, but goes even further by allowing simultaneously two different growth factors depicted in $\big(N(t)\big)^{\alpha}$ and $\big[-\ln(1-N(t))\big]^{\beta}$, as well as two different environmental retroaction factors indicated by $\big(1-N(t)\big)^{\gamma}$ and $\big(-\ln N(t)\big)^{\delta}$. Observe now that the growth factor $N(t)$ can be considered as a linear approximation of the growth factor $-\ln(1-N(t))$, but with the latter stimulating more growth than the former. However, equation (1.10) does not contain a closed form analytical solution, unless for some special combinations of the parameters α, β, γ and δ. For more information on other population growth models, cf. Lotka [10], Tsoularis [14] and Tsoularis and Wallace [15].

The EV distributions that arise as solutions to the Verhulst and some generalized Verhulst equations seem to indicate that there is a close connection between population dynamics and forms of extreme stability. This was our motivation to investigate what kind of relationship is indeed present. Therefore, in Section 2, we shall show that the parameter attached to the retroaction factor of some generalized Verhulst equations determines, on its own, and in most situations, which EV distribution for maxima is suitable to model risks of extreme events in population dynamics. Finally, in Section 3, some overall comments are further provided.

2. EXTREME STABILITY IN SOME GENERALIZED VERHULST MODELS

2.1. Some basic facts in extreme value theory

In extreme value theory (EVT) the logistic distribution, which arises as the solution of Verhulst's normalized logistic equation (1.4), is known to be one of three types of max-geometric-stable distributions, the other two being the log-logistic and backward log-logistic distributions (Rachev and Resnick [11]).

Definition 2.1. A distribution function H is a max-geometric-stable distribution if for all $\theta \in (0,1)$, there exist real numbers $a_\theta = a(\theta) > 0$ and $b_\theta = b(\theta)$ such that

$$H(a_\theta x + b_\theta) = \frac{\theta H(x)}{1-(1-\theta)H(x)}\,.$$

Basically, if $\{X_n\}_{n\in\mathbb{N}}$ is a sequence of independent and identically distributed random variables and $X_{N:N} = \max(X_1,\dots,X_N)$ is the random maximum, where N is a geometric random variable of mean $1/\theta$, independent of each X_n, then, as $\theta \to 0$, max-geometric-stable distributions are the only possible non-degenerate limiting distributions for sequences of linearly normalized random maxima $(X_{N:N} - b_\theta)/a_\theta$.

Another well known fact in EVT is that GEV distributions for maxima, defined in (1.9), are the unique max-stable distributions.

Definition 2.2. A distribution function G is a max-stable distribution if for all $n \in \mathbb{N}$, there exist real numbers $\alpha_n = \alpha(n) > 0$ and $\beta_n = \beta(n)$ such that

$$G^n(\alpha_n x + \beta_n) = G(x)\,.$$

In other words, and as $n \to \infty$, max-stable distributions are the only possible non-degenerate limiting distributions for sequences of linearly normalized maxima $(X_{n:n} - b_n)/a_n$, with $X_{n:n} = \max(X_1, \dots, X_n)$, for $\{X_n\}_{n\in\mathbb{N}}$ a sequence of independent and identically distributed random variables (Gnedenko [7]), or more generally, for stationary weakly dependent random variables with distribution function F (Leadbetter *et al.* [9]). If the aforementioned non-degenerate limit exists, we then say that F is in the domain of attraction for maxima of G_ξ, in (1.9).

Initially, in Gnedenko's seminal paper, there appeared three types of max-stable distributions, which can indeed be combined into a single family, the GEV family of distributions for maxima in (1.9). In particular, if $\xi > 0$, we have the so-called Fréchet distribution, if $\xi < 0$, we obtain the Weibull distribution for maxima and if $\xi = 0$, we get the Gumbel distribution, already defined in (1.6), by taking the limit of (1.9) as $\xi \to 0$. The shape parameter ξ in (1.9) is the extreme value index (EVI), a very important parameter associated with extreme events.

Remark 2.1. From the relation between the minimum and the maximum, namely $\min(X_1, \dots, X_n) = -\max(-X_1, \dots, -X_n)$, we get the GEV distribution for minima, defined by $G^*_\xi(x; \lambda, \delta) = 1 - G_\xi(-x; \lambda, \delta)$, with $G_\xi(x; \lambda, \delta)$ given in (1.9).

In Section 1 we mentioned that, when $K = 1$, the GEV distribution for maxima appears as the solution of equation (1.8), which can be regarded as a generalized Verhulst equation. This makes some sense because there is a strong connection between max-geometric-stable and max-stable distributions. More precisely, if G_ξ represents the distribution function of a GEV distribution for maxima, with EVI ξ, and $H = H_\xi$ represents the distribution function of a max-geometric-stable distribution, we have

$$H_\xi(x; \lambda, \delta) = \frac{1}{1 - \ln G_\xi(x; \lambda, \delta)} = \frac{1}{1 + \big(1 + \xi(x-\lambda)/\delta\big)^{-1/\xi}}\,, 1 + \xi(x-\lambda)/\delta > 0,$$

with $(\xi, \lambda, \delta) \in \mathbb{R} \times \mathbb{R} \times \mathbb{R}^+$. Therefore, we have a close relationship between the log-logistic and Fréchet distributions ($\xi > 0$), between the backward log-logistic and Weibull for maxima distributions ($\xi < 0$) and between the logistic and Gumbel distributions ($\xi = 0$).

In the next subsection we shall be particularly interested in investigating which EV distribution is adequate to model risks of extreme events in population dynamics, when using some generalized Verhulst models. To this end, we recall one of the first order condition for establishing domains of attraction for maxima (or simply max-domains of attraction). In particular, we shall work with the first order condition given in de Haan [6], which is equivalent to the first order condition given in Gnedenko [7].

We say that a distribution function F belongs to the max-domain of attraction of a GEV distribution G_ξ, and we use the notation $F \in \mathcal{D}_{\mathcal{M}}(G_\xi)$, if, and only if,

$$\lim_{t\to\infty} \frac{U(tx)-U(t)}{a(t)} = \begin{cases} \frac{x^\xi-1}{\xi} & , \ \xi \neq 0 \\ \ln x & , \ \xi = 0 \end{cases}, \quad x>0\,, \tag{2.1}$$

where $U(t) = F^{\leftarrow}\left(1-\frac{1}{t}\right)$, $t \geq 1$, is the reciprocal tail quantile function, $F^{\leftarrow}(y) = \inf\{x : F(x) \geq y\}$ is the generalized inverse function of F and $a(.)$ is an adequate positive function.

If $\xi \neq 0$, sometimes it is more convenient to consider the following conditions instead, which are equivalent to (2.1):

a) If $\xi > 0$, we can choose $a(t) = U(t)$, in (2.1), and then $F \in \mathcal{D}_{\mathcal{M}}(G_{\xi>0})$ if, and only if, $\lim\limits_{t\to\infty} \frac{U(tx)}{U(t)} = x^\xi$ for $x>0$;

b) If $\xi < 0$, $U(\infty) < \infty$ and $\lim\limits_{t\to\infty} \frac{U(\infty)-U(t)}{a(t)} = -1/\xi$: Then $F \in \mathcal{D}_{\mathcal{M}}(G_{\xi<0})$ if, and only if, $\lim\limits_{t\to\infty} \frac{U(\infty)-U(tx)}{U(\infty)-U(t)} = x^\xi$, for $x>0$.

Remark 2.2. In spite of the close connection between max-geometric-stable and max-stable distributions, the former class does have their own set of characterizations for domains of attraction. However, $H_\xi \in \mathcal{D}_{\mathcal{M}}(G_\xi)$. Indeed, and with the notation $\overline{F} = 1-F$ for the right tail function, $\overline{H}_\xi = -\ln G_\xi/(1-\ln G_\xi) \sim \overline{G}_\xi/(1-\ln G_\xi)$.

2.2. EV distributions in generalized Verhulst models

Let us consider again the Blumberg hiperlogistic equation

$$\frac{\mathrm{d}}{\mathrm{d}t} N(t) = \rho \left(N(t)\right)^\alpha \left(1-N(t)\right)^\beta\,, \quad \alpha,\beta>0\,. \tag{2.2}$$

Henceforth, we shall assume that $K=1$ in all differential equations in order to get a normalized solution, *i.e.* a distribution function N.

If $\alpha \notin \mathbb{N}$, the solution of (2.2) satisfies the equation

$$\frac{(N(t))^{1-\alpha}}{1-\alpha}\, {}_2F_1\big(1-\alpha,\beta;2-\alpha;N(t)\big) = \rho t + C\,, \tag{2.3}$$

where ${}_2F_1(a,b;c;z) = \sum_{n=0}^{\infty} \frac{(a)_n(b)_n}{(c)_n} \frac{z^n}{n!}$, with $(x)_n = x(x+1)\cdots(x+n-1)$, is the hypergeometric function and C is a real number. Without loss of generality, we can assume that $C=0$.

Since ${}_2F_1(a,b;b;z)=(1-z)^{-a}$, it follows that if $\alpha+\beta=2$ ($\beta=2-\alpha$), we get the closed form analytical solution

$$N(t)=\frac{1}{1+\big((1-\alpha)\rho t\big)^{-1/(1-\alpha)}}=\frac{1}{1+\Big(1+(1-\alpha)\Big(\rho t-\frac{1}{1-\alpha}\Big)\Big)^{-1/(1-\alpha)}}$$

for (2.2), which belongs to the max-geometric-stable family of distributions, with an EVI $\xi=1-\alpha=\beta-1$. Consequently, $N\in\mathcal{D}_{\mathcal{M}}(G_{\xi=1-\alpha=\beta-1})$ (see **Remark 2.2**).

Remark 2.3. Observe that the Verhulst logistic equation (1.2), which is just the Blumberg hiperlogistic equation in (2.2) for $\alpha=\beta=1$, satisfies the condition $\alpha+\beta=2$, assumed in Theorem 2.1, with $\xi=1-\alpha=\beta-1=0$.

We shall next generalize the result above on the basis of the reciprocal tail quantile function associated with the solution that comes out of (2.3) when $\alpha\notin\mathbb{N}$, which is given by

$$U(t)=N^{\leftarrow}\big(1-\tfrac{1}{t}\big)=\frac{1}{\rho}\left(\frac{1}{1-\alpha}\big(1-\tfrac{1}{t}\big)^{1-\alpha}\,{}_2F_1\big(1-\alpha,\beta;2-\alpha;1-\tfrac{1}{t}\big)-C\right).$$

Hence, if $\beta<1$, we have $U(\infty)<\infty$ and if $\beta\geq 1$, $U(\infty)=\infty$. These results follow from the properties of the hypergeometric function, namely ${}_2F_1(a,b;c;1)<\infty$ if $a+b-c<0$ and ${}_2F_1(a,b;c;1)=\infty$ if $a+b-c\geq 0$.

We first state:

Theorem 2.1. *If $\alpha\notin\mathbb{N}$ in the Blumberg hiperlogistic equation* (2.2), *then $N\in\mathcal{D}_{\mathcal{M}}(G_{\xi=\beta-1})$.*

Proof: a) For $\beta<1$,

$$\begin{aligned}\lim_{t\to\infty}&\frac{U(\infty)-U(tx)}{U(\infty)-U(t)}=\\&=\lim_{t\to\infty}\frac{{}_2F_1(1-\alpha,\beta;2-\alpha;1)-\big(1-\frac{1}{tx}\big)^{1-\alpha}\,{}_2F_1\big(1-\alpha,\beta;2-\alpha;1-\frac{1}{tx}\big)}{{}_2F_1(1-\alpha,\beta;2-\alpha;1)-\big(1-\frac{1}{t}\big)^{1-\alpha}\,{}_2F_1\big(1-\alpha,\beta;2-\alpha;1-\frac{1}{t}\big)}\\&=x^{\beta-1}\lim_{t\to\infty}\left(\frac{1-\frac{1}{tx}}{1-\frac{1}{t}}\right)^{-\alpha}=x^{\beta-1}\,.\end{aligned}$$

Therefore, $N\in\mathcal{D}_{\mathcal{M}}(G_{\xi=\beta-1<0})$. To obtain the limit above we took into consideration the fact that

$$\frac{\partial}{\partial t}\big(1-\tfrac{1}{t}\big)^{1-\alpha}\,{}_2F_1\big(1-\alpha,\beta;2-\alpha;1-\tfrac{1}{t}\big)=(1-\alpha)\,\big(1-\tfrac{1}{t}\big)^{-\alpha}\,t^{\beta-2}\,.$$

b) If $\beta > 1$,

$$\lim_{t\to\infty} \frac{U(tx)}{U(t)} = \lim_{t\to\infty} \frac{\frac{1}{1-\alpha}\left(1-\frac{1}{tx}\right)^{1-\alpha} {}_2F_1\left(1-\alpha,\beta;2-\alpha;1-\frac{1}{tx}\right) - C}{\frac{1}{1-\alpha}\left(1-\frac{1}{t}\right)^{1-\alpha} {}_2F_1\left(1-\alpha,\beta;2-\alpha;1-\frac{1}{t}\right) - C}$$
$$= x^{\beta-1} \lim_{t\to\infty} \left(\frac{1-\frac{1}{tx}}{1-\frac{1}{t}}\right)^{-\alpha} = x^{\beta-1}\,.$$

Consequently, $N \in \mathcal{D}_{\mathcal{M}}(G_{\xi=\beta-1>0})$.

c) If $\beta = 1$,

$$\lim_{t\to\infty} \big(U(tx) - U(t)\big) = \frac{1}{(1-\alpha)\rho} \Big[\left(1-\tfrac{1}{tx}\right)^{1-\alpha} {}_2F_1\left(1-\alpha,1;2-\alpha,1-\tfrac{1}{tx}\right) -$$
$$- \left(1-\tfrac{1}{t}\right)^{1-\alpha} {}_2F_1\left(1-\alpha,1;2-\alpha,1-\tfrac{1}{t}\right)\Big]$$
$$= \frac{\ln x}{\rho}\,.$$

Thus, if we consider $a(t) = 1/\rho > 0$, we have

$$\lim_{t\to\infty} \frac{U(tx)-U(t)}{a(t)} = \ln x\,,$$

which means that $N \in \mathcal{D}_{\mathcal{M}}(G_{\xi=\beta-1=0})$.

Note 2.1. The previous limit was obtained with the help of the software Mathematica, since there are series expansions involved and the use of relations between contiguous hypergeometric functions.

□

We next state:

Theorem 2.2. *If $\alpha,\beta \in \mathbb{N}$ in the Blumberg hiperlogistic equation* (2.2), *then we also get $N \in \mathcal{D}_{\mathcal{M}}(G_{\xi=\beta-1})$.*

Proof: a) If $\alpha = n = 2,3,\ldots$ and $\beta = 1$, the solution of (2.2) satisfies now the equation

$$\sum_{k=2}^{n} \frac{1}{1-k}\frac{1}{(N(t))^{k-1}} + \ln\!\left(\frac{N(t)}{1-N(t)}\right) = \rho t + C.$$

Hence, the reciprocal tail quantile function associated with the solution is, in this case,

$$U(t) = \frac{1}{\rho}\left(\sum_{k=2}^{n} \frac{1}{1-k}\frac{1}{\left(1-\frac{1}{t}\right)^{k-1}} + \ln(t-1) - C\right)\,,$$

with $U(\infty)=\infty$. It is quite straightforward to prove that

$$\lim_{t\to\infty}\frac{U(tx)-U(t)}{1/\rho}=\ln x\,,$$

from which follows that $N\in\mathcal{D}_{\mathcal{M}}(G_{\xi=\beta-1=0})$.

b) When $\alpha=1$ and $\beta=m=2,3,\ldots$, we have a solution satisfying the equation

$$\sum_{k=2}^{m}\frac{1}{k-1}\frac{1}{(1-N(t))^{k-1}}+\ln\left(\frac{N(t)}{1-N(t)}\right)=\rho t+C\,,$$

which in turn yields the reciprocal tail quantile function

$$U(t)=\frac{1}{\rho}\left(\sum_{k=2}^{m}\frac{1}{k-1}t^{k-1}+\ln(t-1)-C\right)\,,$$

with $U(\infty)=\infty$. It is also quite straightforward to prove that

$$\lim_{t\to\infty}\frac{U(tx)}{U(t)}=x^{m-1}\,,$$

which means that $N\in\mathcal{D}_{\mathcal{M}}(G_{\xi=\beta-1>0})$.

c) For the more general case $\alpha=n=2,3,\ldots$ and $\beta=m=2,3\ldots$, we get a solution that verifies the equation

$$\sum_{k=2}^{n}\frac{a_k}{1-k}\frac{1}{(N(t))^{k-1}}+\sum_{j=2}^{m}\frac{b_j}{j-1}\frac{1}{(1-N(t))^{j-1}}+A\ln\left(\frac{N(t)}{1-N(t)}\right)=\rho t+C\,,$$

where the a_k and b_j's are real numbers and $A>0$. The reciprocal tail quantile function is now defined by

$$U(t)=\frac{1}{\rho}\left(\sum_{k=2}^{n}\frac{a_k}{1-k}\frac{1}{\left(1-\frac{1}{t}\right)^{k-1}}+\sum_{j=2}^{m}\frac{b_j}{j-1}t^{j-1}+A\ln(t-1)-C\right)\,,$$

with $U(\infty)=\infty$. It easily follows that

$$\lim_{t\to\infty}\frac{U(tx)}{U(t)}=x^{m-1}\,,$$

which means, once again, that $N\in\mathcal{D}_{\mathcal{M}}(G_{\xi=\beta-1>0})$.

□

Remark 2.4. All previous results lead us to conjecture that for all $\alpha,\beta>0$, the solution of equation (2.2) will be in the max-domain of attraction of a GEV distribution with an EVI $\xi=\beta-1$, where β is the retroaction parameter. However, the case $\alpha=2,3,\ldots$ and $\beta\notin\mathbb{N}$ is still left to be proved.

Unfortunately, we cannot use equation (2.3) because the hypergeometric function diverges for the parameters involved. So far we were not able to obtain a general equation as the ones obtained for different scenarios of $\alpha, \beta \in \mathbb{N}$. Nevertheless, the results above, and all particular cases we have tried, indicate that we should have a solution $N \in \mathcal{D}_{\mathcal{M}}(G_{\xi=\beta-1})$. This seems very likely, since it holds true for $\beta \in \mathbb{N}$ and there is no apparent reason why it should not also hold for $\beta \notin \mathbb{N}$.

For example, if $\alpha = 2$ and $\beta = 1/2$, the solution of (2.2) satisfies the equation

$$-\frac{\sqrt{1-N(t)}}{N(t)} - \operatorname{arctanh}\left(\sqrt{1-N(t)}\right) = \rho t + C\,,$$

which yields the reciprocal tail quantile function

$$U(t) = -\frac{1}{\rho}\left(\frac{\sqrt{\frac{1}{t}}}{1-\frac{1}{t}} + \operatorname{arctanh}\left(\sqrt{\tfrac{1}{t}}\right) + C\right)\,,$$

with $U(\infty) < \infty$. Given that

$$\lim_{t\to\infty} \frac{U(\infty) - U(tx)}{U(\infty) - U(t)} = x^{-1/2} = x^{1/2-1}\,,$$

we have $N \in \mathcal{D}_{\mathcal{M}}(G_{\xi=-1/2})$. On the other hand, if, for instance, $\alpha = 3$ and $\beta = 3/2$, the solution of (2.2) satisfies now the equation

$$\frac{-2 - 5N(t) + 15(N(t))^2}{4(N(t))^2\sqrt{1-N(t)}} - \frac{15}{4}\operatorname{arctanh}\left(\sqrt{1-N(t)}\right) = \rho t + C\,,$$

from which the reciprocal tail quantile function is

$$U(t) = \frac{1}{\rho}\left(\frac{-2 - 5\left(1-\frac{1}{t}\right) + 15\left(1-\frac{1}{t}\right)^2}{4\left(1-\frac{1}{t}\right)^2\sqrt{\frac{1}{t}}} - \frac{15}{4}\operatorname{arctanh}\left(\sqrt{\tfrac{1}{t}}\right) - C\right)\,,$$

with $U(\infty) = \infty$. Since

$$\lim_{t\to\infty} \frac{U(tx)}{U(t)} = x^{1/2} = x^{3/2-1}\,,$$

we conclude that $N \in \mathcal{D}_{\mathcal{M}}(G_{\xi=1/2})$.

Remark 2.5. In Blumberg's hiperlogistic equation (2.2) we are not considering the possibility of an absent growth or retroaction factor, *i.e.* $\alpha = 0$ or $\beta = 0$. For example, if we assume that $\alpha = 0$ in (2.2), it is interesting to see that the solution is (for $C = 0$)

$$N(t) = 1 - \left((\beta-1)\rho t\right)^{-1/(\beta-1)} = 1 - \left(1 + (\beta-1)\left(\rho t - \frac{1}{\beta-1}\right)\right)^{-1/(\beta-1)}\,,$$

which is a member of the generalized Pareto (GP) family of distributions with shape parameter $\beta - 1$.

The GP family of distributions has the functional form

$$F_\xi(x;\lambda,\delta) = 1 - \big(1+\xi(x-\lambda)/\delta\big)^{-1/\xi}\,, \quad 1+\xi(x-\lambda)/\delta > 0\,,\ x>\lambda\,, \tag{2.4}$$

with $(\xi,\lambda,\delta) \in \mathbb{R}\times\mathbb{R}\times\mathbb{R}^+$. Once more, ξ, λ and δ are shape, location and scale parameters, respectively. The GP family defined in (2.4) combines into a single family three families of distributions, namely the exponential family, which is the limiting case of (2.4) as $\xi \to 0$, the classical Pareto family ($\xi > 0$) and the so-called Pareto type II family ($\xi < 0$). Note that the uniform distribution, which is the solution of equation (2.2) for $\alpha = \beta = 0$, is a GP distribution when $\xi = -1$.

In EVT, GP distributions also play an important role, more precisely, in modeling peaks over high thresholds. In fact, if X is a random variable with distribution function F, GP distributions arise as the limiting distribution for the distribution of conditional excesses $X - u|X > u$, as $u \to x^F$, where $x^F = \sup\{x : F(x) < 1\}$ is the right endpoint of the underlying model F.

In a population dynamics context what matters to know is that $F_\xi \in \mathcal{D}_\mathcal{M}(G_\xi)$. Therefore, when dealing with the case $\alpha = 0$ in equation (2.2), we have a solution $N \in \mathcal{D}_\mathcal{M}(G_{\xi=\beta-1})$. Note also that if $\beta = 0$ in (2.2), the solution is now

$$N(t) = \big((\alpha-1)(-\rho t)\big)^{-1/(\alpha-1)}\,,$$

which is of the type $1 - F_\xi(-x,\lambda,\delta)$, with F_ξ defined in (2.4), and reminding us of the relation between GEV distributions for minima and for maxima, namely $G^*_\xi(x;\lambda,\delta) = 1 - G_\xi(-x;\lambda,\delta)$. In particular, if $\alpha = 1$, we get as solution $N(t) = \exp(\rho t)$, *i.e.* an exponential growth.

Let us next consider the differential equation

$$\frac{\mathrm{d}}{\mathrm{d}t}N(t) = \rho\,\big(N(t)\big)^\alpha\,\big(-\ln N(t)\big)^\beta\,, \quad \alpha,\beta > 0\,, \tag{2.5}$$

which generalizes equation (1.8) considered in Brilhante *et al.* [4]. We have now the validity of the following:

Theorem 2.3. *If N is the solution of the differential equation* (2.5), *then $N \in \mathcal{D}_\mathcal{M}(G_{\xi=\beta-1})$.*

Proof: If $\alpha = 1$ (and $\beta > 0$), we get the closed form analytical solution (for $C = 0$),

$$N(t) = \exp\Big(-\big((\beta-1)\rho t\big)^{-1/(\beta-1)}\Big) = \exp\bigg(-\Big(1+(\beta-1)\Big(\rho t - \tfrac{1}{\beta-1}\Big)\Big)^{-1/(\beta-1)}\bigg)\,,$$

which is a GEV distribution for maxima with an EVI $\xi = \beta$–1, *i.e.* $N \in \mathcal{D}_\mathcal{M}(G_{\xi=\beta-1})$.

On the other hand, if $\alpha \neq 1$ and:

a) $\beta < 1$, the solution satisfies the equation

$$(1-\alpha)^{\beta-1}\Gamma\big(1-\beta,(\alpha-1)\ln N(t)\big) = \rho t + C\,, \tag{2.6}$$

where $\Gamma(a,z) = \int_z^\infty t^{a-1}\mathrm{e}^{-t}\mathrm{d}t$, $a>0$, is the incomplete gamma function, or equivalently, the solution satisfies the equation

$$-(1-\alpha)^{\beta-1}\gamma(1-\beta,(\alpha-1)\ln N(t)) = \rho t + C\,, \tag{2.7}$$

where $\gamma(a,z) = \int_0^z t^{a-1}\mathrm{e}^{-t}\mathrm{d}t = \Gamma(a) - \Gamma(a,z)$ is another type of incomplete gamma function and $\Gamma(a) = \int_0^\infty t^{a-1}\mathrm{e}^{-t}\mathrm{d}t$, $\alpha > 0$, is the (complete) gamma function.

The reciprocal tail quantile function associated with (2.6) is

$$U(t) = \frac{1}{\rho}\Big((1-\alpha)^{\beta-1}\Gamma\left(1-\beta,(\alpha-1)\ln\big(1-\tfrac{1}{t}\big)\right) - C\Big)\,,$$

with $U(\infty) < \infty$. In this case,

$$\lim_{t\to\infty}\frac{U(\infty)-U(tx)}{U(\infty)-U(t)} = \frac{1}{x}\lim_{t\to\infty}\left(\frac{\ln(1-\frac{1}{tx})}{\ln(1-\frac{1}{t})}\right)^{-\beta}\left(\frac{t-\frac{1}{x}}{t-1}\right)^{-\alpha} = x^{\beta-1}\,,$$

since

$$\frac{\partial}{\partial t}\Gamma\Big(1-\beta,(\alpha-1)\ln\left(1-\tfrac{1}{t}\right)\Big) = -\frac{(\alpha-1)^{1-\beta}\left(1-\frac{1}{t}\right)^{-\alpha}\left(\ln\left(1-\frac{1}{t}\right)\right)^{-\beta}}{t^2}\,.$$

Therefore, we have $N \in \mathcal{D}_{\mathcal{M}}(G_{\xi=\beta-1<0})$.

b) $\beta > 1$, the solution satisfies now the equation

$$\frac{(-\ln N(t))^{1-\beta}}{\beta-1}\,{}_1F_1\big(1-\beta,2-\beta;(1-\alpha)\ln N(t)\big) = \rho t + C\,,$$

where ${}_1F_1(a,b;z) = \sum_{n=0}^\infty \frac{(a)_n}{(b)_n}\frac{z^n}{n!}$ is the confluent hypergeometric function. In this case we are using equation (2.7) because if $a<0$, $\gamma(a,z) = \frac{z^a}{a}\,{}_1F_1(a,a+1;-z)$.

The reciprocal tail quantile function is

$$U(t) = \frac{1}{\rho}\left(\frac{\left(-\ln\left(1-\frac{1}{t}\right)\right)^{1-\beta}}{\beta-1}\,{}_1F_1\big(1-\beta,2-\beta;(1-\alpha)\ln(1-\tfrac{1}{t})\big) - C\right)\,,$$

with $U(\infty) = \infty$, since we have ${}_1F_1(a,b;0) = 1$. Therefore, without loss of generality, if $C=0$,

$$\begin{aligned}\lim_{t\to\infty}\frac{U(tx)}{U(t)} &= \lim_{t\to\infty}\left(\frac{\ln\left(1-\frac{1}{t}\right)}{\ln\left(1-\frac{1}{tx}\right)}\right)^{\beta-1}\frac{{}_1F_1\big(1-\beta,2-\beta;(1-\alpha)\ln(1-\frac{1}{tx})\big)}{{}_1F_1\big(1-\beta,2-\beta;(1-\alpha)\ln(1-\frac{1}{t})\big)}\\ &= x^{\beta-1}\,.\end{aligned}$$

Hence, $N \in \mathcal{D}_{\mathcal{M}}(G_{\xi=\beta-1>0})$.

c) $\beta = 1$, the solution satisfies the equation

$$-\mathrm{Ei}\big((1-\alpha)\ln N(t)\big) = \rho t + C\,,$$

where $\mathrm{Ei}(x) = -\int_{-x}^{\infty} \frac{\mathrm{e}^{-t}}{t}\mathrm{d}t$, $x > 0$, is the exponential integral function. Thus, the reciprocal tail quantile function is

$$U(t) = -\frac{1}{\rho}\Big(\mathrm{Ei}\big((1-\alpha)\ln N(t)\big) + C\Big)\,,$$

with $U(\infty) = \infty$. Since the exponential integral function series expansion is

$$\mathrm{Ei}(x) = \gamma + \ln x + \sum_{n=1}^{\infty} \frac{x^n}{n!n}\,,$$

where $\gamma = 0.57721\ldots$ is Euler's constant (cf. Abramowitz and Stegun [1]), it follows that

$$\lim_{t\to\infty} \big(U(tx) - U(t)\big) = \frac{1}{\rho} \lim_{t\to\infty} \ln\left(\frac{\ln(1-\frac{1}{t})}{\ln(1-\frac{1}{tx})}\right) = \frac{\ln x}{\rho}\,.$$

Therefore,

$$\lim_{t\to\infty} \frac{U(tx) - U(t)}{1/\rho} = \ln x\,,$$

meaning that $N \in \mathcal{D}_{\mathcal{M}}(G_{\xi=\beta-1=0})$.

□

As proved above, the retroaction parameter β of the generalized Verhulst equation (2.5) is the only parameter that establishes which GEV distribution for maxima is adequate to model the risk of extreme events in population dynamics, with the EVI being equal to $\beta - 1$. We saw earlier that, for a large variety of situations, this also happens to be the case when using equation (2.2). Now, this might seem at first sight a bit strange, in the sense that the growth parameter α has no involvement whatsoever in establishing the limit distribution. However, this apparent "abnormality" can be explained by noticing that we are working with normalized equations, and therefore getting normalized solutions, meaning that $N(t) \in (0,1)$. In light of this, we have $(1 - N(t))^{\beta} \to 0$ as $\beta \to \infty$, which in this context is translated into a weaker control on population growth, and therefore the possibility of occurrence of more extreme events. This situation will also be mirrored in the case of working with the retroaction factor $(-\ln N(t))^{\beta}$.

Remark 2.6. It is also interesting to see that the solution of the subfamily of models defined in (1.5) for $\nu > 0$, *i.e.*

$$\frac{\mathrm{d}}{\mathrm{d}t} N(t) = \rho\, N(t) \frac{1 - (N(t))^{\nu}}{\nu}\,, \tag{2.8}$$

and which contains the Verhulst logistic equation as a special case ($\nu = 1$), belongs to the max-domain of attraction of a GEV distribution, with EVI $\xi = 0$. In fact, we already know that the solution for $\nu = 1$ is a member of the logistic family of distributions, and which in turn belongs to the max-domain of attraction of a GEV distribution with EVI $\xi = 0$. But, more generally, the solution of (2.8) satisfies the equation

$$\nu \ln N(t) - \ln\big(1 - (N(t))^\nu\big) = \rho t + C\,,$$

and therefore the reciprocal tail quantile function associated is

$$U(t) = \frac{1}{\rho}\left[\nu \ln\left(1 - \frac{1}{t}\right) - \ln\left(1 - \left(1 - \frac{1}{t}\right)^{\nu}\right) - C\right],$$

with $U(\infty) = \infty$. Since

$$\lim_{t\to\infty} \frac{U(tx) - U(t)}{1/\rho} = \ln x\,,$$

it follows that $N \in \mathcal{D}_{\mathcal{M}}(G_{\xi=0})$.

3. COMMENTS AND FURTHER RESULTS

As mentioned in Section 1, $N(t)$ is a linear approximation of $-\ln(1 - N(t))$, with $N(t) \in (0,1)$. So a valid question is, what happens if $N(t)$ is replaced by $-\ln(1 - N(t))$ in (2.2)? In other words, what kind of solution do we get for the generalized Verhulst equation

$$\frac{\mathrm{d}}{\mathrm{d}t}N(t) = \rho\,\big[-\ln(1 - N(t))\big]^{\alpha}(1 - N(t))^{\beta}\,, \quad \alpha, \beta > 0\,? \tag{3.1}$$

What happens is that the roles between β and α are switched, in the sense that now the growth parameter α establishes, on its own, which EV distribution for minima, not for maxima, is at stake.

In fact, if $\beta = 1$, the solution of (3.1) is a GEV distribution for minima G^*_ξ, with $\xi = \alpha - 1$. As an immediate consequence of the close connection between maxima and minima, there are only three types of stable distributions for minima, namely the Fréchet for minima ($\xi > 0$), the Weibull ($\xi < 0$) and the Gumbel for minima ($\xi = 0$). On the other hand, if $\beta \neq 1$, the solution of (3.1) will belong to the min-domain of attraction of a G^*_ξ, with $\xi = \alpha - 1$. Note that in this new setting we can still have uncontrolled population growth, although this growth will be somehow restricted to minimum levels, due to "lack of space" to accommodate more explosive population growths.

An interesting and open topic of research lies now on the estimation of β on the basis of the estimation of ξ, or the other way round, the estimation of

ξ on the basis of the estimation of β. (For the estimation of ξ, see the recent overview on statistical EVT by Gomes and Guillou [8], among others.) In fact, an adequate estimation of the parameter β is fundamental so that the generalized Verhulst models considered here can be applicable to real data. This is the case, since we have established that the retroaction parameter β is the only parameter that determines which GEV distribution for maxima is appropriate to model the risk of extreme large events in population dynamics, with the EVI (for maxima) being equal to $\beta - 1$. A similar comment applies to the growth parameter α and the modeling of extreme small events in population dynamics, with the EVI for minima being then equal to $\alpha - 1$. Tsoularis and Wallace [15] investigated how the inflection point of the population growth curve is related to its malthusian growth and retroaction parameters, and their results may be exploited in this context.

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