PROOF OF CONJECTURES ON THE STANDARD DEVIATION, SKEWNESS AND KURTOSIS OF THE SHIFTED GOMPERTZ DISTRIBUTION

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Abstract:

- Three conjectures on the standard deviation, skewness and kurtosis of the shifted Gompertz distribution, as the shape parameter increases to $+\infty$, are proved. In this regard, the exponential integral function and polygamma functions are used in the proofs. In addition, an explicit expression for the ith moment of this probabilistic model is obtained. These results allow to place the shifted Gompertz distribution in the Skewness–Kurtosis diagram, providing a valuable help in the decision to choose the shifted Gompertz distribution among the models to fit data. Their usefulness is illustrated by fitting a real malaria data set using the maximum likelihood method for estimating the parameters of the shifted Gompertz distribution and some classical models. Goodness-of-fit measures are used to compare their performance.

Key-Words:

- *Shifted Gompertz distribution; Moment generating function; Skewness–Kurtosis diagram; Polygamma functions.*

AMS Subject Classification:

- 60E05, 62E15, 26B20.

1. INTRODUCTION

One of the most important references in models of adopting timing of innovations is the model of Bass [1]. From this model, Bemmaor [3] formulated that the individual-level model of adopting timing of a new product in a market is randomly distributed according to the shifted Gompertz distribution. More recently, Lover *et al.* [6] show that modeling studies of period of time to first relapse in human infections with malaria in the New World tropical region, can support the shifted Gompertz distribution.

Some statistical properties of the shifted Gompertz distribution were obtained in Bemmaor [3]. Jiménez Torres and Jodrá [8] gave explicit expressions for the first and second moment, a closed form expression for the quantile function was derived, and the limit distributions of extreme order statistics were considered.

In Jiménez Torres [7] the method of least squares, method of maximum likelihood and method of moments to estimate the parameters of the shifted Gompertz distribution were used. In this paper we want to expand and complete the knowledge and statistical properties of the shifted Gompertz distribution, solving the three conjectures presented in Jiménez Torres and Jodrá [8] and obtaining a general expression for the moments.

Although the Gompertz distribution Z has been given in different forms in the literature, the cumulative distribution function (cdf) $F_Z(z) = P(Z \leq z) = e^{-\alpha e^{-\beta z}}$, $-\infty < z < +\infty$, found in Bemmaor [3], satisfies that its standard deviation, skewness and excess kurtosis are equals to $\pi/(\sqrt{6}\beta)$, $12\sqrt{6}\zeta(3)/\pi^3$ and 2.4, respectively, where $\zeta(\cdot)$ denotes the Riemann zeta function. The skewness of a random variable X is defined by $\gamma_1 = E[(X-\mu)^3]/\sigma^3$ and is a measure of the asymmetry of the probability distribution. The excess kurtosis of X is given by $\gamma_2 = E[(X-\mu)^4]/\sigma^4 - 3$ and it describes the shape of the tails of the probability distribution.

Let X be a random variable having the shifted Gompertz distribution with parameters α and β, where $\alpha > 0$ is a shape parameter and $\beta > 0$ is a scale parameter. The probability density function of X is

$$f_X(x) = \beta e^{-(\beta x + \alpha e^{-\beta x})}\big(1 + \alpha\big(1 - e^{-\beta x}\big)\big) \qquad x > 0. \tag{1.1}$$

This model can be characterized as the maximum of two independent random variables with Gompertz distribution (parameters $\alpha > 0$ and $\beta > 0$) and exponential distribution (parameter $\beta > 0$). From (1.1), given that $\lim_{\alpha\to 0} f_X(x) = \beta e^{-\beta x}$, it may be noted that the shifted Gompertz distribution gets close to an exponential distribution with mean $1/\beta$, as the parameter α decreases to 0. So, for a fixed value of β, $\lim_{\alpha\to 0}\sigma = 1/\beta$, where σ is the standard deviation of X. For the shifted Gompertz distribution we have $\lim_{\alpha\to 0}\gamma_1 = 2$ and $\lim_{\alpha\to 0}\gamma_2 = 6$, which are the skewness and kurtosis of the exponential distribution. If the shape parameter α increases to infinity, the asymptotic behavior of the shifted Gompertz distribution is nontrivial and these limits require analytic tools for their calculation.

Based on numerical evidence showed in Jiménez Torres and Jodrá [8] the next three conjectures were presented:

$$\text{Conjecture } 1: \qquad \lim_{\alpha\to+\infty} \sigma = \frac{\pi}{\sqrt{6}\beta}$$

$$\text{Conjecture } 2: \qquad \lim_{\alpha\to+\infty} \gamma_1 = \frac{12\sqrt{6}\zeta(3)}{\pi^3}$$

$$\text{Conjecture } 3: \qquad \lim_{\alpha\to+\infty} \gamma_2 = 2.4$$

The remainder of this note is organized as follows. In Section 2, we prove Conjecture 1. In Section 3, we provide an explicit expression for the ith moment of the shifted Gompertz distribution. In Section 4 and Section 5, we prove Conjecture 2 and Conjecture 3, respectively. In Section 6 we show the importance of these results in the choice of the shifted Gompertz distribution among the models to fit a real data set and finally, the main conclusions are presented in Section 7.

2. PROOF OF CONJECTURE 1

In Jiménez Torres and Jodrá [8] explicit expressions for the moments of orders 1 and 2 of X were obtained. The first moment of X, or mean μ of X, is

$$E[X] = \frac{1}{\beta}\left(\gamma + \log(\alpha) + E_1(\alpha) + \frac{1-e^{-\alpha}}{\alpha}\right), \tag{2.1}$$

where $\gamma \approx 0.57721$ is the Euler–Mascheroni constant and $E_1(x)$ is the exponential integral function, defined by $E_1(x) = \int_x^{+\infty} \frac{e^{-t}}{t} dt$, $x > 0$. The second moment of X is

$$E[X^2] = \frac{2}{\alpha\beta^2}\Big(\gamma + \log(\alpha) + E_1(\alpha) + {}_3F_3[1,1,1;2,2,2;-\alpha]\alpha^2\Big), \tag{2.2}$$

where ${}_3F_3[1,1,1;2,2,2;-\alpha] = \sum_{k=1}^{+\infty} \frac{(-\alpha)^{k-1}}{k!k^2}$ is a generalized hypergeometric function. Moreover, we need the next expression (see Geller and Ng [5]) for $a > 0$ and $b > 0$

$$\int_b^{+\infty} \frac{E_1(ax)}{x} dx = \frac{1}{2}\big((\gamma + \log(ab))^2 + \zeta(2)\big) + \sum_{k=1}^{+\infty} \frac{(-ab)^k}{k!k^2}, \tag{2.3}$$

where $\zeta(2) = \dfrac{\pi^2}{6}$. In particular, using (2.3) with $a = 1$ and $b = \alpha$, we obtain

$$\int_{\alpha}^{+\infty} \frac{E_1(x)}{x} dx = \frac{1}{2}\big((\gamma + \log(\alpha))^2 + \zeta(2)\big) + \sum_{k=1}^{+\infty} \frac{(-\alpha)^k}{k!k^2}, \tag{2.4}$$

and in the next theorem, we prove Conjecture 1.

Theorem 2.1. *The limit of the standard deviation, σ, of the shifted Gompertz distribution X as the shape parameter α increases to $+\infty$ is finite and its value is*

$$\lim_{\alpha\to+\infty} \sigma = \frac{\pi}{\sqrt{6}\beta}. \tag{2.5}$$

Proof: The variance of a random variable X is $\sigma^2 = E[X^2] - (E[X])^2$. From (2.1), (2.2) and (2.4) we have

$$\begin{aligned}
\sigma^2 &= \frac{2}{\alpha\beta^2}\left[\gamma + \log(\alpha) + E_1(\alpha) - \alpha\int_{\alpha}^{+\infty}\frac{E_1(x)}{x}dx + \frac{\alpha}{2}\big((\gamma+\log(\alpha))^2 + \zeta(2)\big)\right] \\
&\quad - \frac{1}{\beta^2}\left(\gamma + \log(\alpha) + E_1(\alpha) + \frac{1-e^{-\alpha}}{\alpha}\right)^2 \qquad (2.6)\\
&= \frac{\zeta(2)}{\beta^2} + R(\alpha),
\end{aligned}$$

where

$$\begin{aligned}
R(\alpha) &= \frac{2}{\alpha\beta^2}\big(\gamma + \log(\alpha) + E_1(\alpha)\big) - \frac{2}{\beta^2}\int_{\alpha}^{+\infty}\frac{E_1(x)}{x}dx \\
&\quad + \frac{1}{\beta^2}\big((\gamma+\log(\alpha)\big)^2 - \frac{1}{\beta^2}\left(\gamma + \log(\alpha) + E_1(\alpha) + \frac{1-e^{-\alpha}}{\alpha}\right)^2.
\end{aligned} \tag{2.7}$$

So, $\lim\limits_{\alpha\to+\infty} \sigma^2 = \zeta(2)/\beta^2 + \lim\limits_{\alpha\to+\infty} R(\alpha)$. Now, in (2.7) we take limit as α increases to $+\infty$, taking into account the next limits related to the exponential integral function (see Geller and Ng [5]):

$$\lim_{x\to+\infty}\big(\log(x)E_1(x)\big) = \lim_{x\to+\infty}\big(e^{-x}E_1(x)\big) = \lim_{x\to+\infty}\big(x^p E_1(x)\big) = 0. \tag{2.8}$$

So, $\lim\limits_{\alpha\to+\infty} R(\alpha) = 0$, and Conjecture 1 is proved. $\square$

To prove Conjecture 2 and Conjecture 3 we need expressions of the moments of orders 3 and 4, respectively. In Section 3 we are more ambitious and obtain a general expression for the moment of order i of the shifted Gompertz distribution.

3. MOMENT OF ORDER i OF X

The ith moment of X, denoted and defined by $E[X^i] = \int_0^{+\infty} x^i f_X(x)dx$, $i = 1, 2, \ldots$, where $f_X(x)$ is given in (1.1), does not seem to have a closed-form expression in terms of elementary functions, but we can find a series expansion. Let $\gamma(a, b)$ be the lower incomplete gamma function defined for any $a > 0$ and $b > 0$ by

$$\gamma(a,b) = \int_0^b v^{a-1} e^{-v} dv, \tag{3.1}$$

and let $M_X(t)$ be the moment generating function of X, i.e., $M_X(t) = E\big[e^{tX}\big]$. In the next theorem we obtain an expression of this function.

Theorem 3.1. *The moment generating function of the shifted Gompertz distribution X for $|t| < \beta$ is*

$$M_X(t) = \alpha^{t/\beta-1}(\alpha + t/\beta)\gamma(1 - t/\beta, \alpha) + e^{-\alpha}. \tag{3.2}$$

Proof: By definition, we have

$$\begin{aligned} E\big[e^{tX}\big] &= \int_0^{+\infty} e^{tx} f_X(x)dx = \beta \int_0^{+\infty} e^{tx-\beta x-\alpha e^{-\beta x}}(1 + \alpha(1 - e^{-\beta x}))dx \\ &= (1+\alpha)\beta \int_0^{+\infty} e^{tx-\beta x-\alpha e^{-\beta x}} dx - \alpha\beta \int_0^{+\infty} e^{tx-2\beta x-\alpha e^{-\beta x}} dx. \end{aligned} \tag{3.3}$$

The change of variable $v = \alpha e^{-\beta x}$ in (3.3) provides

$$\begin{aligned} E\big[e^{tX}\big] &= \alpha^{t/\beta-1}(1+\alpha) \int_0^{\alpha} e^{-v} v^{-t/\beta} dv - \alpha^{t/\beta-1} \int_0^{\alpha} e^{-v} v^{1-t/\beta} dv \\ &= \alpha^{t/\beta-1}\big((1+\alpha)\gamma(1 - t/\beta, \alpha) - \gamma(2 - t/\beta, \alpha)\big). \end{aligned} \tag{3.4}$$

Integrating by parts in (3.1) yields the recurrence relation $\gamma(a+1, b) = a\gamma(a, b) - b^a e^{-b}$. So, we have

$$E\big[e^{tX}\big] = \alpha^{t/\beta-1}\big((\alpha + t/\beta)\gamma(1 - t/\beta, \alpha) + \alpha^{1-t/\beta} e^{-\alpha}\big), \tag{3.5}$$

thereby completing the proof. □

According to Theorem 3.1, the moment generating function of the shifted Gompertz distribution, $M_X(t)$, is finite in the open neighborhood $(-\beta, \beta)$ of 0. In particular, it implies that moments of all orders exist. In the next result, we provide an explicit expression of the moment of order i.

Theorem 3.2. *The moment of order i, $i = 1, 2, \ldots$, of the shifted Gompertz distribution X is*

$$E[X^i] = \frac{i!}{\beta^i}\left(1 + \sum_{k=1}^{+\infty}\left(\frac{1}{(k+1)^i} - \frac{1}{k^i}\right)\frac{(-\alpha)^k}{k!}\right). \tag{3.6}$$

Proof: Since $M_X(t)$ is finite for t in $(-\beta, \beta)$, it can be expanded in a Taylor series about 0 and the moments of X can be computed by differentiation of $M_X(t)$ at $t = 0$, i.e., $M_X^{(i)}(t)|_{t=0} = M_X^{(i)}(0) = E[X^i]$, $i = 1, 2, \ldots$, where $M_X^{(i)}(t)$ denotes the ith derivative of the moment generating function of X. That is,

$$M_X(t) = 1 + \sum_{i=1}^{+\infty}\frac{E[X^i]}{i!}t^i \qquad |t| < \beta. \tag{3.7}$$

Given the Taylor series of the exponential function e^{-v} in (3.1), we have the following series expansion of the lower incomplete gamma function

$$\gamma(a, b) = \int_0^b \sum_{k=0}^{+\infty}(-1)^k\frac{v^{a+k-1}}{k!}dv = \sum_{k=0}^{+\infty}\frac{(-1)^k b^{a+k}}{(a+k)k!}. \tag{3.8}$$

From (3.8), we have

$$\gamma(1 - t/\beta, \alpha) = \sum_{k=0}^{+\infty}\frac{(-1)^k\alpha^{1-t/\beta+k}}{(1 - t/\beta + k)k!} \qquad |t| < \beta, \tag{3.9}$$

and substituting (3.9) in (3.2), we obtain

$$M_X(t) = (\alpha + t/\beta)\sum_{k=0}^{+\infty}\frac{(-1)^k\alpha^k}{(1 - t/\beta + k)k!} + e^{-\alpha} \qquad |t| < \beta. \tag{3.10}$$

But the real number $(1 - t/\beta + k)^{-1}$ can be expressed as the sum of the terms of a geometric series, i.e.,

$$\frac{1}{1 - t/\beta + k} = \frac{1}{k+1}\sum_{i=0}^{+\infty}\frac{t^i}{(k+1)^i\beta^i} \qquad |t| < \beta. \tag{3.11}$$

Finally, substituting (3.11) in (3.10),

$$M_X(t) = (\alpha + t/\beta)\sum_{k=0}^{+\infty}\frac{(-\alpha)^k}{(k+1)!}\sum_{i=0}^{+\infty}\frac{t^i}{(k+1)^i\beta^i} + e^{-\alpha} \qquad |t| < \beta. \tag{3.12}$$

Identifying term to term of (3.7) and (3.12), we have

$$E[X^i] = \frac{i!}{\beta^i}\left(1 - \sum_{k=1}^{+\infty}\left(\frac{1}{k!k^i} - \frac{1}{(k+1)!(k+1)^{i-1}}\right)(-\alpha)^k\right), \tag{3.13}$$

thereby completing the proof of Theorem 3.2. □

4. PROOF OF CONJECTURE 2

To prove Conjecture 2 we need the next expression (see Geller [4]) for $a > 0$ and $\rho > 0$

$$\begin{aligned}\int_0^\rho e^{-ax}\log^3(x)dx &= -6\rho\left(\sum_{k=0}^{+\infty}\frac{(-a\rho)^k}{k!(k+1)^4} - \log(\rho)\sum_{k=0}^{+\infty}\frac{(-a\rho)^k}{k!(k+1)^3}\right)\\ &\quad -\frac{3}{a}\log^2(\rho)\Big(\gamma + \log(a\rho) + E_1(a\rho) - \frac{1}{3}\log(\rho)(1-e^{-a\rho})\Big).\end{aligned} \tag{4.1}$$

It may be noted that (4.1) corrects one misprint in Geller [4] (the sign of $\frac{1}{3}\log(\rho)(1-e^{-a\rho})$). In particular, using (4.1) with $a = 1$ and $\rho = \alpha$, we have

$$\begin{aligned}\int_0^\alpha e^{-x}\log^3(x)dx &= -6\alpha\left(\sum_{k=0}^{+\infty}\frac{(-\alpha)^k}{k!(k+1)^4} - \log(\alpha)\sum_{k=0}^{+\infty}\frac{(-\alpha)^k}{k!(k+1)^3}\right)\\ &\quad -3\log^2(\alpha)\Big(\gamma + \log(\alpha) + E_1(\alpha) - \frac{1}{3}\log(\alpha)(1-e^{-\alpha})\Big).\end{aligned} \tag{4.2}$$

Moreover, we need the value of (4.2) as α increases to $+\infty$, i.e., $\int_0^{+\infty} e^{-x}\log^3(x)dx$. This integral es $\Gamma^{(3)}(1)$, the third derivative of gamma function evaluated at 1, where the gamma function is defined by $\Gamma(p) = \int_0^{+\infty} t^{p-1}e^{-t}dt$, for a real number $p > 0$. To know the value of $\Gamma^{(3)}(1)$ we can use the digamma function, define by $\psi(p) = \Gamma'(p)/\Gamma(p)$ and polygamma functions, $\psi'(p)$, $\psi^{(2)}(p)$, $\psi^{(3)}(p)$ …. These functions are derivatives of the logarithm of the gamma function. In particular, we have $\psi(1) = -\gamma$ and $\psi^{(n)}(1) = (-1)^{n+1}n!\zeta(n+1)$, for $n = 1, 2, 3\ldots$ (see, e.g. [9, 5.15.2]). Using this relation we have $\psi'(1) = \zeta(2)$ and $\psi^{(2)}(1) = -2\zeta(3)$. So, the value of $\Gamma^{(3)}(1)$ is

$$\Gamma^{(3)}(1) = (\psi(1))^3 + 3\psi(1)\psi'(1) + \psi^{(2)}(1) = -\gamma^3 - 3\gamma\zeta(2) - 2\zeta(3), \tag{4.3}$$

where $\zeta(3) \approx 1.20205$ is a real number known as Apéry's constant. In the next theorem, we prove Conjecture 2.

Theorem 4.1. *The limit of the coefficient of skewness, γ_1, of the shifted Gompertz distribution X as the shape parameter α increases to $+\infty$ is finite and its value is*

$$\lim_{\alpha\to+\infty}\gamma_1 = \frac{12\sqrt{6}\zeta(3)}{\pi^3}. \tag{4.4}$$

Proof: The coefficient of skewness of X is

$$\gamma_1 = E[(X-\mu)^3]/\sigma^3 = \big(E[X^3] - 3\mu E[X^2] + 2\mu^3\big)/\sigma^3. \tag{4.5}$$

We can study every term of this equation. The first term of (4.5) is $E[X^3]$. According to (3.6), the moment of order 3 of X is

$$E[X^3] = \frac{3!}{\beta^3}\left(1 + \sum_{k=1}^{+\infty}\left(\frac{1}{(k+1)^3} - \frac{1}{k^3}\right)\frac{(-\alpha)^k}{k!}\right). \tag{4.6}$$

From (4.2), we have

$$\begin{aligned}\sum_{k=0}^{+\infty}\frac{(-\alpha)^k}{k!(k+1)^4} &= -\frac{1}{6\alpha}\bigg(\int_0^\alpha e^{-x}\log^3(x)dx - 6\alpha\log(\alpha)\sum_{k=0}^{+\infty}\frac{(-\alpha)^k}{k!(k+1)^3}\\ &\quad +3\log^2(\alpha)\big(\gamma+\log(\alpha)+E_1(\alpha) - \frac{1}{3}\log(\alpha)(1-e^{-\alpha})\big)\bigg).\end{aligned} \tag{4.7}$$

Given that $\sum_{k=1}^{+\infty}\frac{(-\alpha)^k}{k!k^i} = -\alpha\sum_{k=0}^{+\infty}\frac{(-\alpha)^k}{k!(k+1)^{i+1}}$, $i = 0, 1, 2, \ldots$, from (2.4), (4.6) and (4.7)

$$\begin{aligned}E[X^3] &= -\frac{1}{\beta^3}\bigg[6\big(\alpha^{-1}+\log(\alpha)\big)\bigg(\int_\alpha^{+\infty}\frac{E_1(x)}{x}dx - \frac{1}{2}\big((\gamma+\log(\alpha))^2+\zeta(2)\big)\bigg)\\ &\quad + \int_0^\alpha e^{-x}\log^3(x)dx + 3\log^2(\alpha)\big(\gamma+\log(\alpha)+E_1(\alpha)\big) - \log^3(\alpha)(1-e^{-\alpha})\bigg].\end{aligned} \tag{4.8}$$

Now, we study $-3\mu E[X^2]$, the second term of (4.5). From (2.1) and (2.2), it is

$$\begin{aligned}-3\mu E[X^2] &= -\frac{6}{\alpha\beta^3}\left(\gamma+\log(\alpha)+E_1(\alpha)+\frac{1-e^{-\alpha}}{\alpha}\right)\\ &\quad\times\left(\gamma+\log(\alpha)+E_1(\alpha) - \alpha\sum_{k=1}^{+\infty}\frac{(-\alpha)^k}{k!k^2}\right),\end{aligned} \tag{4.9}$$

and from (2.4), we have

$$\begin{aligned}-3\mu E[X^2] &= -\frac{6}{\alpha\beta^3}\left(\gamma+\log(\alpha)+E_1(\alpha)+\frac{1-e^{-\alpha}}{\alpha}\right)\\ &\quad\times\left(\gamma+\log(\alpha)+E_1(\alpha) - \alpha\int_\alpha^{+\infty}\frac{E_1(x)}{x}dx + \frac{\alpha}{2}\big((\gamma+\log(\alpha))^2+\zeta(2)\big)\right).\end{aligned} \tag{4.10}$$

The third term of (4.5) is $2\mu^3$. From (2.1), it is

$$2\mu^3 = \frac{2}{\beta^3}\left(\gamma + \log(\alpha) + E_1(\alpha) + \frac{1-e^{-\alpha}}{\alpha}\right)^3. \tag{4.11}$$

Finally, taking into account the three terms of (4.5), i.e., (4.8), (4.10) and (4.11), that $\lim_{\alpha\to+\infty}\int_0^\alpha e^{-x}\log^3(x)dx = \Gamma^{(3)}(1)$ given in (4.3) and the limits (2.8), we have

$$\lim_{\alpha\to+\infty} E[(X-\mu)^3] = \frac{2\zeta(3)}{\beta^3}. \tag{4.12}$$

According to Theorem 2.1, $\lim_{\alpha\to+\infty}\sigma^3 = \dfrac{\pi^3}{6\sqrt{6}\beta^3}$, and Conjecture 2 is proved. $\square$

5. Proof of Conjecture 3

To prove Conjecture 3 we need the next expression (see Geller [4]), valid for $a>0$, $\rho>0$, $p>-1$ and $n=0,1,2,3,\ldots$

$$\int_0^\rho x^p e^{-ax}\log^n(x)dx = (-1)^n n!\rho^{p+1}\sum_{k=0}^{n}\frac{(-1)^k\log^k(\rho)}{k!}\sum_{l=0}^{+\infty}\frac{(-a\rho)^l}{l!(p+l+1)^{n-k+1}}. \tag{5.1}$$

In particular, we need (5.1) for $a=1$, $\rho=\alpha$, $p=0$ and $n=4$, i.e.,

$$\begin{aligned}\int_0^\alpha e^{-x}\log^4(x)dx &= 4!\alpha\sum_{k=0}^{4}\frac{(-1)^k\log^k(\alpha)}{k!}\sum_{l=0}^{+\infty}\frac{(-\alpha)^l}{l!(l+1)^{5-k}}\\ &= 4!\alpha\left[\sum_{k=0}^{+\infty}\frac{(-\alpha)^k}{k!(k+1)^5} - \log(\alpha)\sum_{k=0}^{+\infty}\frac{(-\alpha)^k}{k!(k+1)^4}\right.\\ &\quad + \frac{\log^2(\alpha)}{2}\sum_{k=0}^{+\infty}\frac{(-\alpha)^k}{k!(k+1)^3} - \frac{\log^3(\alpha)}{3!}\sum_{k=0}^{+\infty}\frac{(-\alpha)^k}{k!(k+1)^2}\\ &\quad \left. + \frac{\log^4(\alpha)}{4!}\sum_{k=0}^{+\infty}\frac{(-\alpha)^k}{k!(k+1)}\right].\end{aligned} \tag{5.2}$$

Moreover, we need the value of (5.2) as α increases to $+\infty$, i.e., $\int_0^{+\infty} e^{-x}\log^4(x)dx$, the $4th$ Euler–Mascheroni integral. This integral is $\Gamma^{(4)}(1)$, the fourth derivative of $\Gamma(p)$, evaluated at $p=1$. Given that $\psi^{(3)}(1) = 6\zeta(4)$, $\big(\zeta(2)\big)^2 = 5\zeta(4)/2$ and $\zeta(4) = \pi^4/90$, the value of $\Gamma^{(4)}(1)$ is

$$\begin{aligned}\Gamma^{(4)}(1) &= \big(\psi(1)\big)^4 + 6\psi'(1)\big(\psi(1)\big)^2 + 4\psi^{(2)}(1)\psi(1) + \psi^{(3)}(1) + 3\big(\psi'(1)\big)^2\\ &= \gamma^4 + 6\gamma^2\zeta(2) + 8\gamma\zeta(3) + \frac{27}{2}\zeta(4).\end{aligned} \tag{5.3}$$

In the next theorem, we prove Conjecture 3.

Theorem 5.1. *The limit of the excess kurtosis, γ_2, of the shifted Gompertz distribution X as the shape parameter α increases to $+\infty$ is finite and its value is*

$$\lim_{\alpha\to+\infty}\gamma_2 = 2.4. \tag{5.4}$$

Proof: The excess kurtosis of X is

$$\gamma_2 = E[(X-\mu)^4]/\sigma^4 - 3 = (E[X^4] - 4\mu E[X^3] + 6\mu^2 E[X^2] - 3\mu^4)/\sigma^4 - 3. \tag{5.5}$$

We can study every term of this equation. The first term of (5.5) is $E[X^4]$. According to (3.6), the fourth moment of X is

$$\begin{aligned} E[X^4] &= \frac{4!}{\beta^4}\Big(1+\sum_{k=1}^{+\infty}\Big(\frac{1}{(k+1)^4}-\frac{1}{k^4}\Big)\frac{(-\alpha)^k}{k!}\Big) \\ &= \frac{4!}{\alpha\beta^4}\Big(\alpha\sum_{k=1}^{+\infty}\frac{(-\alpha)^k}{k!(k+1)^4}+\alpha^2\sum_{k=0}^{+\infty}\frac{(-\alpha)^k}{k!(k+1)^5}\Big). \end{aligned} \tag{5.6}$$

From (5.2), we have

$$\begin{aligned} \sum_{k=0}^{+\infty}\frac{(-\alpha)^k}{k!(k+1)^5} &= \frac{1}{24\alpha}\Big[\int_0^\alpha e^{-x}\log^4(x)dx + 24\alpha\log(\alpha)\sum_{k=0}^{+\infty}\frac{(-\alpha)^k}{k!(k+1)^4} \\ &\quad -12\alpha\log^2(\alpha)\sum_{k=0}^{+\infty}\frac{(-\alpha)^k}{k!(k+1)^3} + 4\alpha\log^3(\alpha)\sum_{k=0}^{+\infty}\frac{(-\alpha)^k}{k!(k+1)^2} \\ &\quad -\alpha\log^4(\alpha)\sum_{k=0}^{+\infty}\frac{(-\alpha)^k}{k!(k+1)}\Big]. \end{aligned} \tag{5.7}$$

From (4.7) and (5.7),

$$\begin{aligned} E[X^4] = \frac{24}{\alpha\beta^4}\Big[&-\frac{1}{6}\int_0^\alpha e^{-x}\log^3(x)dx - \frac{1}{2}\log^2(\alpha)\big(\gamma+\log(\alpha)+E_1(\alpha)\big) \\ &-\frac{1}{6}\log^3(\alpha)(1-e^{-\alpha}) - \log(\alpha)\Big(\int_\alpha^{+\infty}\frac{E_1(x)}{x}dx - \frac{1}{2}\big((\gamma+\log(\alpha)\big)^2 \\ &+\zeta(2)\big)\Big) + \frac{\alpha}{24}\Big(\int_0^\alpha e^{-x}\log^4(x)dx - 4\log(\alpha)\int_0^\alpha e^{-x}\log^3(x)dx \\ &-12\log^2(\alpha)\int_\alpha^{+\infty}\frac{E_1(x)}{x}dx + 6\log^2(\alpha)\big((\gamma+\log(\alpha))^2+\zeta(2)\big) \\ &-8\log^3(\alpha)\big(\gamma+\log(\alpha)+E_1(\alpha)\big) + 3\log^4(\alpha)(1-e^{-\alpha})\Big]. \end{aligned} \tag{5.8}$$

Now, we study $-4\mu E[X^3]$, the second term of (5.5). From (2.1) and (4.8), it is

$$\begin{aligned}
-4\mu E[X^3] &= -\frac{4}{\beta^4}\left(\gamma+\log(\alpha)+E_1(\alpha)+\frac{1-e^{-\alpha}}{\alpha}\right) \\
&\times\Bigg[-6\big(\alpha^{-1}+\log(\alpha)\big)\left(\int_\alpha^{+\infty}\frac{E_1(x)}{x}dx-\frac{1}{2}\big((\gamma+\log(\alpha))^2+\zeta(2)\big)\right) \\
&-\int_0^\alpha e^{-x}\log^3(x)dx-3\log^2(\alpha)\big(\gamma+\log(\alpha)+E_1(\alpha)\big)+\log^3(\alpha)(1-e^{-\alpha})\big)\Bigg].
\end{aligned} \tag{5.9}$$

The third term of (5.5) is $6\mu^2E[X^2]$. From (2.1), (2.2) and (2.4), it is

$$\begin{aligned}
6\mu^2E[X^2] &= \frac{12}{\beta^4}\left(\gamma+\log(\alpha)+E_1(\alpha)+\frac{1-e^{-\alpha}}{\alpha}\right)^2 \\
&\times\left[\frac{\gamma+\log(\alpha)+E_1(\alpha)}{\alpha}-\int_\alpha^{+\infty}\frac{E_1(x)}{x}dx+\frac{1}{2}\big((\gamma+\log(\alpha))^2+\zeta(2)\big)\right].
\end{aligned} \tag{5.10}$$

The fourth and last term of (5.5) is $-3\mu^4$. From (2.1), it is

$$-3\mu^4=-\frac{3}{\beta^4}\left(\gamma+\log(\alpha)+E_1(\alpha)+\frac{1-e^{-\alpha}}{\alpha}\right)^4. \tag{5.11}$$

Finally, taking into account that $\lim_{\alpha\to+\infty}\int_0^\alpha e^{-x}\log^4(x)dx=\Gamma^{(4)}(1)$ given in (5.3), the four terms of (5.5), i.e., (5.8), (5.9), (5.10) and (5.11), and the limits (2.8), we have

$$\lim_{\alpha\to+\infty}E[(X-\mu)^4]=\frac{27\zeta(4)}{2\beta^4}. \tag{5.12}$$

According to Theorem 2.1, $\lim_{\alpha\to+\infty}\sigma^4=\frac{\pi^4}{36\beta^4}$. Given the value of $\zeta(4)=\frac{\pi^4}{90}$, Conjecture 3 is proved. □

6. REAL DATA APPLICATION

One of the human malaria parasites with the widest geographic distribution in the world is *plasmodium vivax*. If a patient was not fully cured or insufficiently treated, he can relapse in a few weeks after the initial infection, i.e., new clinical symptoms begin after the disease disappeared from the blood following the primary infection. In this section, we have considered an application with periods of time to first relapse or recurrence in 38 patients located at Brazil. We have chosen Brazil since it is located geographically in the New World tropical region, where following Lover *et al.* [6], the shifted Gompertz distribution is suitable for modeling times to first relapse. Tropical region is delimited by the ±23.5° latitude lines. Table 1 shows times (days) to first relapse observed, reported in Battle *et al.* [2].

31	32	32	33	34	35	37	37	44	45	48	53	57	57	58	62	63	64	68
69	70	70	70	71	75	78	80	82	83	86	91	97	97	112	124	132	158	185

Table 1: Real data set: Times (days) to first relapse observed (malaria parasite *plasmodium vivax*) in 38 patients located at Brazil.

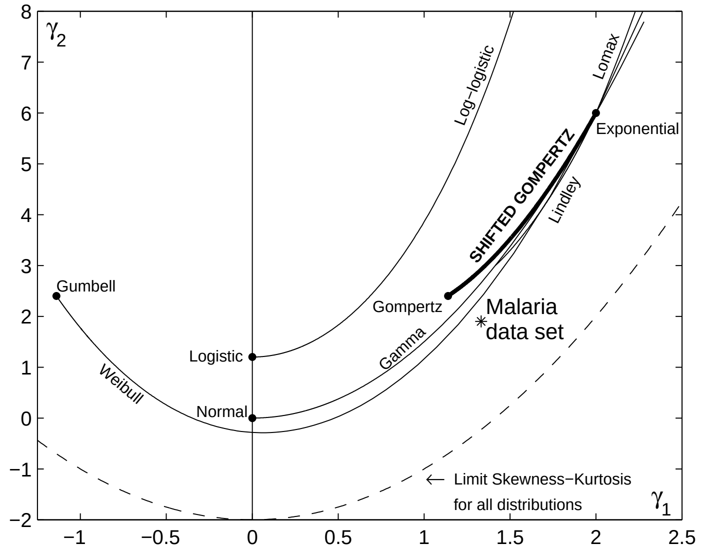

Figure 1: Skewness (γ_1) versus excess kurtosis (γ_2) for some probabilistic models and the locus of the malaria data set.

According to Theorem 4.1, the values of γ_1 of the shifted Gompertz distribution are greater than $12\sqrt{6}\zeta(3)/\pi^3 \approx 1.1395$, i.e., are always positive and possibly this can be a good model to fit a data set with positive asymmetry. Similarly, according to Theorem 5.1, the values of γ_2 of the shifted Gompertz distribution are greater than 2.4, i.e., are always positive. This means that the shifted Gompertz distribution is a fat-tailed probability distribution, and possibly it can be a good model to fit a data set with positive excess kurtosis.

The results proved in this paper allow to place the shifted Gompertz distribution in the Skewness–Kurtosis diagram (see Vargo *et al.* [11]). This moment-ratio diagram (see Figure 1) is a plot containing the (γ_1, γ_2) values for probability distributions. When a probabilistic model has no shape parameter (for example, normal, logistic, Gompertz, exponential or Gumbell distribution, among other), its locus in this diagram corresponds to a point. When a probabilistic model has one shape parameter (for example, log-logistic, gamma, Weibull, Lindley, Lomax or shifted Gompertz distribution, among other), its locus in this diagram corresponds to a curve. In this diagram, the shifted Gompertz distribution starts at the locus of the exponential distribution and ends at the locus of Gompertz distribution. Also, in Figure 1 there is a curve representing the frontier $\gamma_2 \geq \gamma_1^2 - 2$ for all distributions (see Stuart and Ord [10]).

Given the observed values of skewness and excess kurtosis for malaria data set ($\gamma_1 = 1.3317$, $\gamma_2 = 1.9009$), we can place it in this diagram (see Figure 1) and use it as valuable help in model selection (see chosen models in Table 2).

Model	Shape parameter	$F(x)$
Exponential: E(λ)	-	$1 - e^{-\lambda x}$
Gamma: G(α, β)	α	$\gamma(\alpha, x/\beta)/\Gamma(\alpha)$
Gompertz: GO(α, β)	-	$e^{-\alpha e^{-\beta x}}$
Gumbell: GU(α, β)	-	$1 - e^{-\alpha e^{\beta x}}$
Lindley: LD(θ)	θ	$1 - (1 + \theta + \theta x)e^{-\theta x}/(1 + \theta)$
Logistic: LG(μ, s)	-	$(1 + e^{-(x-\mu)/s})^{-1}$
Log-logistic: LL(λ, p)	p	$1 - (1 + (\lambda x)^p)^{-1}$
Lomax: LO(α, β)	α	$1 - (1 + \beta x)^{-\alpha}$
Normal: N(μ, σ)	-	$\Phi((x - \mu)/\sigma)$
Weibull: W(α, β)	α	$1 - e^{-(x/\beta)^\alpha}$
Shifted Gompertz: SG(α, β)	α	$e^{-\alpha e^{-\beta x}}(1 - e^{-\beta x})$

Table 2: Models and their cumulative distribution functions $F(x)$.

Model	MLE parameter		$-$LogL	AIC	BIC	K-S	p-val(K-S)	W*	A*
Malaria data set									
E	0.0139	-	200.290	402.580	404.218	0.351	10^{-4}	1.113	5.749
G	4.965	14.414	183.079	370.158	373.434	**0.089**	**0.923**	0.054	0.448
GO	10.024	0.040	182.983	369.967	373.242	0.102	0.823	0.047	0.425
GU	0.126	0.022	198.302	400.605	403.880	0.226	0.041	0.492	2.899
LD	0.0275	-	189.961	381.923	383.561	0.220	0.049	0.386	2.359
LG	67.520	18.091	186.779	377.558	380.834	0.117	0.673	0.056	0.667
LL	0.015	3.821	221.181	446.363	449.638	0.103	0.811	1.606	8.397
LO	0.114	99.634	278.846	561.693	564.968	0.600	10^{-12}	3.477	16.062
N	71.578	34.505	188.481	380.963	384.238	0.138	0.461	0.169	1.156
W	2.202	81.129	185.522	375.045	378.320	0.113	0.714	0.107	0.779
SG	8.709	0.040	**182.759**	**369.518**	**372.793**	0.101	0.831	**0.046**	**0.419**

Best fitting model is shown in bold.

Table 3: The MLEs of the parameters and goodness-of-fit tests.

It is reasonable to think that models located relatively near the locus of malaria data set (for example, Weibull, gamma, Gompertz or shifted Gompertz distribution) can provide a better fit than models located farther away (for example, Gumbell, logistic, normal or Lomax, among other). To accept or rejected this surmise, we estimate the parameters of the shifted Gompertz distribution and of all models represented in Figure 1 by the maximum likelihood method. We obtain the performance of each model based on the following goodness-of-fit measures: log-likelihood function (LogL), Akaike Information Criterion (AIC), Bayesian Information Criterion (BIC), Kolmogorov–Smirnov (K-S) statistic with the corresponding p-value, Cramer von Mises (W*) and Anderson-Darling (A*).

The results obtained (see Table 3) show that the shifted Gompertz distribution presents the best fit in almost all goodness-of-fit measures. The smallest values of −LogL, AIC, BIC, W* and A* correspond to the shifted Gompertz distributions. The best values of K-S and its p-value are obtained by gamma and shifted Gompertz distribution. In addition, Weibull, gamma or Gompertz distribution present, in general, better fit than Gumbell, logistic, normal, Lindley, exponential, log-logistic or Lomax distribution.

7. Conclusions

Three conjectures on the standard deviation, skewness and kurtosis of the shifted Gompertz distribution, as the shape parameter α increases to $+\infty$, have been proved, solving the asymptotic problems found in Jiménez Torres and Jodrá [8]. In addition, an explicit expression for the ith moment of the shifted Gompertz distribution has been obtained. These results allow to place the shifted Gompertz distribution in the Skewness–Kurtosis diagram, starting at the locus of the exponential distribution and ending at the locus of Gompertz distribution. To check their usefulness, a real malaria data set has been fitted, estimating the parameters by maximum likelihood. The results obtained show that the shifted Gompertz distribution presents a very good fit among the analyzed models, suggesting that the results proved in this paper can play an important rule in the decision to choose this model to fit data.

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REFERENCES

[1] Bass, F.M. (1969). A new product growth model for consumer durables. *Management Science*, **15**, 5:215–227.

[2] Battle, K.E., Karhunen, M.S., Bhatt, S., Gething, P.W., Howes, R.E., Golding, N., Van Boeckel, T.P., Messina, J.P., Shanks, G.D., Smith, D.L., Baird, J.K. and Hay, S.I. (2014). Geographical variation in *plasmodium vivax* relapse. *Malaria Journal*, 13:144.

[3] Bemmaor, A.C. (1994). *Modeling the diffusion of new durable goods: word-of-mouth effect versus consumer heterogeneity.* In "Research Traditions in Marketing" (G. Laurent, G.L. Lilien and B. Pras, Eds.), Boston. MA: Kluwer, 201–223.

[4] Geller, M. (1963). A table of integrals involving powers, exponentials, logarithms, and the exponential integral. *Jet Propulsion Laboratory.* Technical Report No. 32–469.

[5] Geller, M. and Ng, E.W. (1969). A table of integrals of the exponential integral. *Journal of Research of the National Bureau of Standards–B. Mathematics and Mathematical Science*, **73B**, No. 3, July-September.

[6] Lover, A.A., Zhao, X., Gao, Z., Coker, R.L. and Cook, A.R. (2014). The distribution of incubation and relapse times in experimental human infections with the malaria parasite *plasmodium vivax*. *BMC Infectious Diseases*, 14:539.

[7] Jiménez Torres, F. (2014). Estimation of parameters of the shifted Gompertz distribution using least squares, maximum likelihood and moments methods. *Journal of Computational and Applied Mathematics*, **256**:867–877.

[8] Jiménez Torres, F. and Jodrá, P. (2009). A note on the moments and computer generation of the shifted Gompertz distribution. *Communications in Statistics-Theory and Methods*, **38**:75–89.

[9] Olver, F.W.J., Lozier, D.W., Boisvert, R.F. and Clark, C.W. (eds.) (2010). *NIST Handbook of Mathematical Functions*, Cambridge University Press.

[10] Stuart, A. and Ord, K. (1994). *Kendall's Advanced Theory of Statistics. Volume I*, NY: Hodder Arnold, New York.

[11] Vargo, E., Pasupathy, R. and Leemis, L.M. (2010). Moment-ratio diagrams for univariate distributions. *Journal of Quality Technology*, **42**, 3:1–11.