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A CHARACTERIZATION OF THE GENERALIZED BIRNBAUM–SAUNDERS DISTRIBUTION

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Abstract:

- The Birnbaum–Saunders (BS) model is a fatigue life distribution related to the normal one with appealing properties. It is a non-negative transformation of a N(0,1) random variable Z, and its extended version, the so-called generalized Birnbaum–Saunders (GBS) model, obtained replacing Z by any other symmetric continuous random variable, shares the same properties. In this paper we characterize the GBS distribution according to one of its outstanding properties and take advantage of this characterization to derive a graphical procedure to assess whether an observed data set follows a GBS distribution. We further develop a goodness-of-fit test for the hypothesis that the data follow a BS distribution with unknown parameters and apply the results to real data sets.

Key-Words:

- *Birnbaum–Saunders distribution; goodness-of-fit; graphical tools; TTT plot.*

AMS Subject Classification:

- 62E10, 63E15, 62-09.

1. INTRODUCTION

Birnbaum and Saunders [10] proposed a fatigue life distribution representing time to failure of materials exposed to a cyclically repeated stress pattern. They obtained a nonnegative transformation of a standard normal random variable (RV) with a scale and a shape parameter known as the Birnbaum–Saunders (BS) distribution; see also Johnson *et al.* [25], pp. 651–663. The BS distribution is positively skewed allowing for different degrees of kurtosis and its hazard or failure rate (FR) has an inverse bathtub shape (see Kundu *et al.* [27] and Bebbington *et al.* [9]). The BS distribution is closed under scale transformations and under reciprocation.

Seshadri [39] and Saunders [38] considered classes of absolutely continuous non-negative RVs closed under reciprocation and ways of generating such classes. Common RVs satisfying the "reciprocal property", meaning that the RV and its own reciprocal are identically distributed, are the Fisher–Snedecor $F_{n,n}$ and the lognormal with appropriate mean, as well as the quotient of two independent and identically distributed (IID) non-negative (and unlimited to the right) RVs, discussed by Gumbel and Keeney [21]. Some other examples of distributions satisfying the reciprocal property can be found in Jones [26] and Vanegas *et al.* [42].

The generalized Birnbaum–Saunders distribution (GBS), introduced by Díaz-García and Leiva [16], is obtained replacing the normal generator in the BS distribution by any symmetric absolutely continuous RV. It is a highly flexible class of positively skewed distributions allowing for a wide range of kurtosis. The probability density functions include unimodal and bimodal cases and FRs can be monotone, inverse bathtub or have more than one change-point. Heavy tails are also allowed depending on the tails of the generating RV. The GBS distribution is also closed under scale transformations and under reciprocation. As a consequence, any GBS distributed RV, suitably scaled, satisfies the reciprocal property. See also Sanhueza *et al.* [36] for a discussion of its theory and applications, Balakrishnan *et al.* [6] for the case generated by scale-mixtures of normal distributions and Leiva *et al.* [30] for a family related to scale-mixture BS distributions.

Among several extensions of the BS distributions that have appeared in the literature, we point out the three-parameter model of power transformations of BS distributed RVs introduced by Owen [34], obtained by relaxing the assumption of independent crack extensions to a long memory process and related to sinh-spherical distributions (see Díaz-García *et al.* [17]) and the four-parameter extension based on the Johnson system (Athayde *et al.* [4]). The BS distribution also belongs to the family of cumulative damage distributions (see Leiva *et al.* [32]).

Truncated BS and shifted BS distributions have been considered as well; see Ahmed *et al.* [2] and Leiva *et al.* [28]. The case generated by non-symmetric RVs has also been addressed; see Ferreira *et al.* [20] and Leiva *et al.* [31]. However in this case the resulting RV does not satisfy the reciprocal property for any scale transformation.

Reparameterizations of the BS model have also appeared; see Leiva *et al.* [29], Santos-Neto *et al.* [37] and references therein.

Considering the problem of fitting a distribution to univariate data and assuming it comes from a nonnegative RV, we would like to assess whether a GBS distribution is a good candidate to model the data, and then find an appropriate way to test for goodness-of-fit (GOF). Notice that the BS and the GBS distributions are not in the location-scale (LS) family. For the case of parametric distributions with unknown parameters, GOF techniques have also been addressed in the literature for non LS models, including graphical techniques; see Barros *et al.* [8] and Castro-Kuriss *et al.* [12] for an overview of available tests and graphical tools to assess GOF in non LS distributions. These techniques can be applied to the case of a BS distribution or a GBS generated by a parameterized distribution, such as the Student or logistic ones, provided a proper estimation procedure and the inverse cumulative distribution function are available, but are not designed to test for the GBS class as a whole.

As mentioned before, any GBS distributed RV, suitably scaled, has the reciprocal property, and we prove that the converse is also true, i.e., any RV that upon a suitable change of scale is equally distributed to its own reciprocal admits a representation as a GBS distribution. This characterization was the starting point that led us to tackle the proposed problem. Namely, it enabled us to find an alternative estimator for the scale parameter, to consider an empirical graphical technique that requires no estimation of the scale parameter and to test whether the data come from a GBS distribution using symmetry tests about an unknown constant. In addition, we consider a test for the null hypothesis that the data come from a BS distribution with unknown parameters and carry out a study of its asymptotic behavior.

The remainder of this paper is organized as follows. In Section 2 we present some well known results about BS and GBS distributions. In Section 3 we establish a characterization of the GBS model related to the reciprocal property and analyze some of its consequences. In Section 4 we discuss the problem of finding out whether this model is suitable to fit a given data set and develop an asymptotic GOF test for the case of the BS distribution with unknown parameters. In Section 5 we apply the results to real data sets and finally in Section 6 we draw some conclusions.

2. BACKGROUND

The BS distribution is a transformation T of a standard normal RV given by

$$T = \beta \left(\frac{\alpha Z}{2} + \sqrt{\left(\frac{\alpha Z}{2}\right)^2 + 1} \right)^2, \tag{2.1}$$

where $Z \sim \mathrm{N}(0,1)$, α $(\alpha > 0)$ is a shape parameter and β $(\beta > 0)$ is a scale parameter (and also the median), denoted here by $T \sim \mathrm{BS}(\alpha,\beta)$, with inverse transformation given by

$$Z = \frac{1}{\alpha} \left(\sqrt{\frac{T}{\beta}} - \sqrt{\frac{\beta}{T}} \right) \sim \mathrm{N}(0,1)\,. \tag{2.2}$$

As mentioned in the previous section, the distribution of T given by (2.1) is positively skewed allowing for different degrees of kurtosis (greater than 3) and its FR has an inverse bathtub shape. Among the properties of this distribution, we highlight that if $T \sim \mathrm{BS}(\alpha,\beta)$ then (i) $cT \sim \mathrm{BS}(\alpha, c\beta)$ with $c > 0$ and (ii) $T^{-1} \sim \mathrm{BS}(\alpha, \beta^{-1})$, i.e., the BS distribution is closed under scale transformations and under reciprocation. Thus, denoting $Y = T/\beta$, Y and $1/Y$ are identically distributed, i.e., Y has the reciprocal property (see Saunders [38]). Analogously we say that $T \sim \mathrm{BS}(\alpha,\beta)$, suitably scaled, satisfies the reciprocal property.

The GBS distribution, introduced by Díaz-García and Leiva [16], is obtained replacing Z in (2.1) by any symmetric absolutely continuous RV X, thus leading to

$$T = \beta \left(\frac{\alpha X}{2} + \sqrt{\left(\frac{\alpha X}{2}\right)^2 + 1} \right)^2, \tag{2.3}$$

where α $(\alpha > 0)$ is a shape parameter and β $(\beta > 0)$ is a scale parameter (and also the median). We say that T given by (2.3) is generated by X and denote it by $T \sim \mathrm{GBS}(\alpha,\beta,g_X)$, where $g_X(\cdot)$ is the probability density function (PDF) of X.

The cumulative distribution function (CDF) of $T \sim \mathrm{GBS}(\alpha,\beta,g_X)$ is given by $F_T(t) = G_X(\xi(t;\alpha,\beta))$, $t > 0$, where $G_X(\cdot)$ is the CDF of X and $\xi(t;\alpha,\beta) = \frac{1}{\alpha}\left(\sqrt{\frac{t}{\beta}} - \sqrt{\frac{\beta}{t}}\right)$, $t > 0$, and the PDF is given by $f_T(t) = g_X(\xi(t;\alpha,\beta))\,\xi'(t;\alpha,\beta)$, $t > 0$, where $\xi'(t;\alpha,\beta) = \frac{t+\beta}{2\alpha\sqrt{\beta}}\, t^{-3/2}$, $t > 0$. The GBS distribution is implemented in the R software package `gbs` (`http://www.R-project.org`).

Once again letting $T \sim \mathrm{GBS}(\alpha, \beta, g_X)$, then T, suitably scaled, satisfies the reciprocal property considering like before $Y = T/\beta$. Notice further that in formula (2.3) we may assume $\alpha = 1$ without loss of generality since letting $X_\alpha = \alpha X$, the distributions $\mathrm{GBS}(\alpha, \beta, g_X)$ and $\mathrm{GBS}(1, \beta, g_{X_\alpha})$ are clearly the same.

The main result in Seshadri [39] states that for absolutely continuous non-negative RVs the reciprocal property is equivalent with the logarithm of that RV being symmetric about zero. In other words, a RV Y satisfies the reciprocal property if and only if $W = \log(Y)$ is symmetric (i.e., W and $-W$ are equally distributed).

3. MAIN RESULTS

As mentioned before, the GBS distribution given by (2.3), where X is a symmetric absolutely continuous RV, is such that T/β and β/T are equally distributed. We prove that the converse is also true, and consequently that this remarkable property characterizes the class of GBS distributions.

Theorem 3.1. *Let T be a non-negative absolutely continuous random variable. Then $T \sim \mathrm{GBS}(\alpha, \beta, g_X)$ if and only if T, suitably scaled, satisfies the reciprocal property.*

Proof: It suffices to prove that if for some β ($\beta > 0$), T/β and β/T are equally distributed RVs then $T \sim \mathrm{GBS}(1, \beta, g_X)$, i.e., T can be written as $T = \beta\left(\frac{X}{2} + \sqrt{\left(\frac{X}{2}\right)^2 + 1}\right)^2$ for some symmetric RV X with PDF $g_X(\cdot)$, where the inverse transformation is $X = \sqrt{\frac{T}{\beta}} - \sqrt{\frac{\beta}{T}}$. So let T be a nonnegative absolutely continuous RV with CDF $H_T(\cdot)$, such that T/β and β/T are equally distributed for some positive constant β. Let $\xi(t) = \xi(t; 1, \beta) = \sqrt{\frac{t}{\beta}} - \sqrt{\frac{\beta}{t}}$ for $t > 0$ and $F(\cdot)$ be the CDF of $X = \xi(T)$, given by $F(z) = H_T(\xi^{-1}(z))$, $z \in \mathbb{R}$, or equivalently $H_T(t) = F(\xi(t))$, $t > 0$. Now, denoting by $F_{\frac{T}{\beta}}(\cdot)$ and $F_{\frac{\beta}{T}}(\cdot)$ the CDFs of T/β and β/T respectively, we then have $F_{\frac{T}{\beta}}(x) = F_{\frac{\beta}{T}}(x) = P\left(\frac{\beta}{T} \leq x\right) = P\left(\frac{T}{\beta} \geq \frac{1}{x}\right) = 1 - F_{\frac{T}{\beta}}(1/x)$, $x > 0$. Consequently $H_T(\beta x) = 1 - H_T\left(\frac{\beta}{x}\right)$ and thus $F(\xi(\beta x)) = 1 - F(\xi(\beta/x))$. From the fact that $\xi(\beta/x) = -\xi(\beta x)$, it follows that $F(\xi(\beta x)) = 1 - F(-\xi(\beta x))$, i.e., $F(z) = 1 - F(-z)$, $z \in \mathbb{R}$. This proves that X is a symmetric RV such that $T = \beta\left(\frac{X}{2} + \sqrt{\left(\frac{X}{2}\right)^2 + 1}\right)^2$, and therefore $T \sim \mathrm{GBS}(1, \beta, g_X)$. $\square$

Corollary 3.1. *Let T be a non-negative absolutely continuous random variable. Then $T \sim \mathrm{GBS}(\alpha, \beta, g_X)$ if and only if $\log(T) - \log(\beta)$ is a symmetric* RV*.*

Remark 3.1. From the characterization in Theorem 3.1, it follows that any class of non-negative absolutely continuous RVs satisfying the reciprocal property belongs to the GBS class. In this case, its median is necessarily 1 ($\beta = 1$). Notice that the support of a GBS distribution can be a proper subset of $[0, +\infty)$. In fact, an example follows directly from Habibullah [23] who introduced a one-parameter family of RVs with support $[a, 1/a]$, where $0 < a < 1$, that satisfy the reciprocal property, and thus belongs to the GBS class. Its CDF is given by $H_a(x;\theta) = \frac{1}{3}\alpha_1(\log x)^3 + \alpha_2 \log x + \frac{1}{2}$, $a < x < \frac{1}{a}$, where $\alpha_1 = \frac{3(1-\theta)}{4(\log a)^3}$, $\alpha_2 = \frac{\theta-3}{4\log a}$ and $0 \le \theta \le 1$. Notice further that if we extend this family by adding a scale parameter β, we get a $\mathrm{GBS}(1, \beta, g_X)$ distribution with CDF given by $H_{a,\beta}(x;\theta) = H_a(x/\beta;\theta)$.

Other results follow immediately from this characterization, as stated in the next three corollaries to Theorem 3.1. Recall that for a random sample $T_1, ..., T_n$ from a GBS distribution, the modified moment (MM) estimator of β is given by

$$\widetilde{\beta} = \sqrt{SR}\,, \tag{3.1}$$

where S and R are the sample arithmetic and harmonic mean, respectively, i.e., $S = \overline{T} = \frac{1}{n}(T_1 + \cdots + T_n)$ and $R^{-1} = \overline{T^{-1}} = \frac{1}{n}\left(\frac{1}{T_1} + \cdots + \frac{1}{T_n}\right)$. See Birnbaum & Saunders [11], Ng *et al.* [33] and Sanhueza *et al.* [36] for the case of BS and GBS distributions. Notice that $\widetilde{\beta}/\beta$ and $\beta/\widetilde{\beta}$ are identically distributed (see Saunders [38], Theorem 3.2).

Corollary 3.2. *Let T and U be two independent* GBS *distributed* RV*s and $a \neq 0$. Then T^a, TU and T/U are also* GBS *distributed* RV*s.*

Corollary 3.3. *Any non-negative* RV *that is written as a quotient of two* IID RV*s is* GBS *distributed.*

Corollary 3.4. *The* MM *estimator of β is* GBS *distributed.*

Notice that Corollary 3.2 states that the GBS class is closed under power transformations, as well as under products and quotients of independent RVs. As an example, any power of $T \sim \mathrm{BS}(\alpha, \beta)$ belongs to the GBS class. Another example (see also Seshadri [39]), following from Corollary 3.3, is that the half-Cauchy distribution with PDF $f(t) = \frac{2}{\pi}\frac{1}{1+t^2}$, $t > 0$, belongs to the GBS class since it is obtained as a quotient of two IID half-normal RVs. Clearly, its logarithm is a symmetric RV, since its PDF is given by $g(x) = e^x f(e^x) = \frac{2}{\pi}\frac{1}{e^x + e^{-x}}$, $x \in \mathbb{R}$.

Remark 3.2. Another consequence of Corollary 3.2 is that any member of the three-parameter extended BS distribution in the sense of Owen [34], consisting of power transformations of BS RVs, given by

$$T = \beta\left(\frac{\alpha Z}{2} + \sqrt{\left(\frac{\alpha Z}{2}\right)^2 + 1}\right)^{\sigma} \qquad \text{or} \qquad Z = \frac{1}{\alpha}\left(\left(\frac{T}{\beta}\right)^{1/\sigma} - \left(\frac{\beta}{T}\right)^{1/\sigma}\right),$$

where $Z \sim \mathrm{N}(0,1)$ and α, β and σ are non-negative parameters, also admits a representation as a $\mathrm{GBS}(1, \beta, g_W)$ for some symmetric absolutely continuous RV W that depends on α, σ and Z. In fact, we have explicitly

$$W = \left(\frac{\alpha Z}{2} + \sqrt{\left(\frac{\alpha Z}{2}\right)^2 + 1}\right)^{\sigma/2} - \left(\frac{\alpha Z}{2} + \sqrt{\left(\frac{\alpha Z}{2}\right)^2 + 1}\right)^{-\sigma/2}.$$

An analogous result holds for the three-parameter extension of GBS distributions referred in Sanhueza *et al.* [36], related to sinh-spherical laws (see Díaz-García *et al.* [17]), obtained replacing Z by a symmetric RV X.

Theorem 3.2. *For a random sample from the* $\mathrm{GBS}(\alpha, \beta, g_X)$ *distribution and assuming* $\mathrm{E}(X^4) < +\infty$*, the* MM *estimator of* β *is asymptotically* $\mathrm{BS}(n^{-1/2}\alpha\theta, \beta)$ *distributed, where* $\theta^2 = \frac{u_1 + \frac{1}{4}\alpha^2 u_2}{1 + \frac{1}{2}\alpha^2 u_1}$*, and* $u_i = \mathrm{E}(X^{2i})$*,* $i = 1, 2$*.*

Proof: The proof is analogous to the proof of Theorem 3.7 in Birnbaum & Saunders [11], replacing Z_i by X_i. In fact, notice that now $1 + \frac{\alpha^2}{2n}\sum X_i^2$ converges in probability to $1 + \frac{1}{2}\alpha^2 u_1$, where $u_i = E(X^{2i})$, and letting again $U_i = X_i(1 + \frac{1}{4}\alpha^2 X_i^2)^{1/2}$, then $\mathrm{var}(U_i) = u_1 + \frac{1}{4}\alpha^2 u_2$. This leads to the limiting distribution $\mathrm{BS}(n^{-1/2}\alpha\theta, \beta)$ with $\theta^2 = \frac{u_1 + \frac{1}{4}\alpha^2 u_2}{1 + \frac{1}{2}\alpha^2 u_1}$. $\square$

Corollary 3.5. *For a random sample of the* $\mathrm{GBS}(\alpha, \beta, g_X)$ *distribution, the* MM *estimator of* β *is asymptotically* $\mathrm{N}(\beta, n^{-1/2}\alpha\theta\beta)$*.*

Remark 3.3. Theorem 3.2 states that the asymptotic GBS distribution for $\widetilde{\beta}$ mentioned in Corollary 3.4 is precisely a $\mathrm{BS}(n^{-1/2}\alpha\theta, \beta)$. It extends the result by Birnbaum & Saunders [11] stating that $\widetilde{\beta}$, in the BS case, is asymptotically distributed as a $\mathrm{BS}(\alpha\theta n^{-1/2}, \beta)$, where $\theta^2 = \frac{4+3\alpha^2}{(2+\alpha^2)^2}$. Moreover, it is in agreement with the more general asymptotic bivariate normal distribution for the MM estimators of α and β (see Ng *et al.* [33] and Sanhueza *et al.* [36]), since a $\mathrm{BS}(\alpha, \beta)$ distribution is asymptotically $\mathrm{N}(\beta, \alpha\beta)$, as $\alpha \to 0$. In fact, this result follows immediately from the power series expansion of (2.1), namely $T = \beta\left(1 + \alpha Z + \frac{1}{2}\alpha^2 Z^2 + \frac{1}{8}\alpha^3 Z^3 - \frac{1}{128}\alpha^5 Z^5 + \cdots\right)$; see Engelhardt *et al.* [19].

4. GOODNESS-OF-FIT

When dealing with a univariate lifetime random sample, $t = (t_1, t_2, ..., t_n)$ from a RV T, a natural question to consider is whether a member of the GBS class is suitable to model these data. Let $Y = \log(T)$, and let t^{-1} and y denote the transformed samples $\left(\frac{1}{t_1}, \frac{1}{t_2}, ..., \frac{1}{t_n}\right)$ and $(\log(t_1), \log(t_2), ..., \log(t_n))$, respectively. Theorem 3.1 and Corollary 3.1 leads us to tackle this problem (i) testing for equal distributions of T/β and β/T with unknown β, or (ii) testing Y for symmetry about an unknown constant. Both these procedures rely on estimating β or $\log(\beta)$, and the same applies from an empirical point of view using a graphical approach such as a quantile-quantile plot (QQ-plot) for the two samples, $\beta^{-1}t$ and βt^{-1} . A graphical procedure known as the total time on test (TTT) plot can also be used and this plot requires no estimation of β.

To test for equal distributions for T/β and β/T, β may be estimated by minimizing some "distance" between the two samples, $\beta^{-1}t$ and βt^{-1}. Two possible distances are:

- The square of the difference between the sample means of $\beta^{-1}t$ and βt^{-1}. This leads to the usual MM estimator of β, given by (3.1) with $S = \overline{T}$ and $R^{-1} = \overline{T^{-1}}$ as before. This is not surprising since the MM estimator of β in the GBS(α, β, f_X) model does not depend on either f_X or α.
- A Kolmogorov–Smirnov (KS) type distance between the empirical CDF (ECDF) of the samples $T_1/\beta, ..., T_n/\beta$ and $\beta/T_1, ..., \beta/T_n$, given by

  $$D_{KS} = \sup_x \left|F_1(x) - F_2(x)\right| ,$$

  where $F_1(x)$ and $F_2(x)$ are the ECDFs of the two samples. Notice that these two samples are not independent.

To test $Y = \log(T)$ for symmetry about unknown location, we highlight two tests with asymptotically distribution-free test statistics, namely (i) a classical test based on the sample skewness coefficient b_1 (Gupta [22]) and (ii) the triples test (see Davis *et al.* [15] and Randles *et al.* [35]). In the first case, the test statistic $\frac{\sqrt{n}\, b_1}{\tau}$, where $b_1 = \frac{m_3}{m_2^{3/2}}$, $\tau = \frac{m_6 - 6m_2m_4 + 9m_2^3}{m_3^2}$ and m_i is the central moment of order i, $i \in \mathbb{N}$, is asymptotically $\mathrm{N}(0, 1)$ under the null hypothesis of symmetry, provided $\mu_6 = \mathrm{E}(Y^6)$ exists. The second test is based on the difference D between the number of "right triples" and the number of "left triples" in the sample, where each triple (Y_i, Y_j, Y_k), $1 \le i < j < k \le n$, is defined as a "right triple" if the middle ordered observation in (Y_i, Y_j, Y_k) is closer to the smallest than to the largest of the three observations, and as a "left triple" if the middle ordered observation is closer to the largest than to the smallest of the three observations.

The test statistic, $V = D/\widehat{\sigma}$, where $\widehat{\sigma}$ is given in formula (3.78) in Hollander *et al.* [24], is asymptotically $\mathrm{N}(0,1)$ under the null hypothesis of symmetry. Notice further that these two tests are unsensitive to power transformations in T, as well as to scale changes.

4.1. A graphical procedure based on the TTT plot

As mentioned in Section 2, the FR is an important indicator in lifetime analysis. Some particular outstanding FR shapes include increasing (IFR), decreasing (DFR), bathtub (BT) and inverse bathtub (IBT) ones. For a RV T with finite expectation, it is possible to identify the shape of its FR by the scaled TTT curve (Barlow *et al.* [7]), given by

$$W_T(y) = \frac{\int_0^{F_T^{-1}(y)} \big[1 - F_T(t)\big]\, dt}{\int_0^{F_T^{-1}(1)} \big[1 - F_T(t)\big]\, dt} , \qquad 0 \le y \le 1 . \tag{4.1}$$

This function can be empirically approximated by

$$W_n(k/n) = \frac{\sum_{i=1}^{k} T_{i:n} + [n-k]\, T_{k:n}}{\sum_{i=1}^{n} T_{i:n}} , \qquad k = 0, ..., n , \tag{4.2}$$

where $T_{1:n}, T_{2:n}, ..., T_{n:n}$ denote the order statistics associated to a random sample $T_1, T_2, ..., T_n$ (see Figure 1). Thus, the plot of $\big[k/n,\, W_n(k/n)\big]$, where the consecutive points are connected by straight lines, gives us information about the underlying FR (see Aarset [1]).

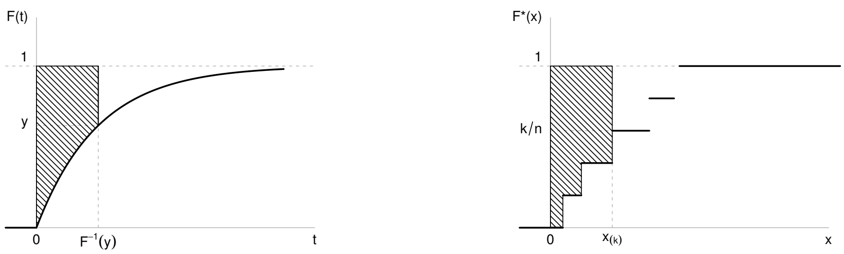

Figure 1: Shaded areas corresponding to $\int_0^{F_T^{-1}(y)} [1 - F_T(t)]\, dt$ (left) and $\frac{1}{n}\big[\sum_{i=1}^{k} T_{i:n} + [n-k]\, T_{k:n}\big]$ (right) in Equations 4.1 and 4.2.

The scaled TTT plot is a straight line in the case of the exponential distribution, a concave (convex) function in the case of an increasing (decreasing) FR, first concave (convex) and then convex (concave) in the case of an inverse bathtub (bathtub) FR, thus providing a useful tool in identifying the shape of the FR (see Figure 2). Further, the scaled TTT is invariant under change of scale, and so in the case of a GBS distribution no estimation of β is required for the plot. The only drawback is that it requires T to have a finite expectation.

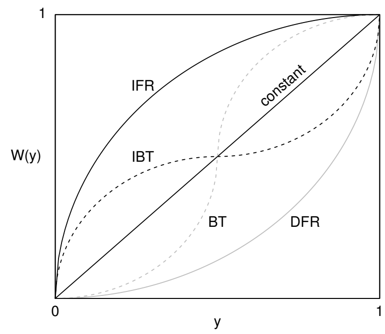

Figure 2: Scaled TTT plot for indicated shape of FR – bathtub (BT), decreasing (DFR), inverse bathtub (IBT), increasing (IFR).

Once again, it follows from Theorem (3.1) that the TTT curves are the same for T/β and β/T if and only if $T \sim \text{GBS}(1, \beta, g_X)$ for some symmetric X. Based on this result, we propose to assess the fit to the GBS distribution by comparing the empirical scaled TTTs of the samples t and t^{-1}. If the data do follow a GBS distribution, these two plots should look alike, regardless of β. We denote by D_{TTT} the maximum vertical distance between these two scaled TTT plots.

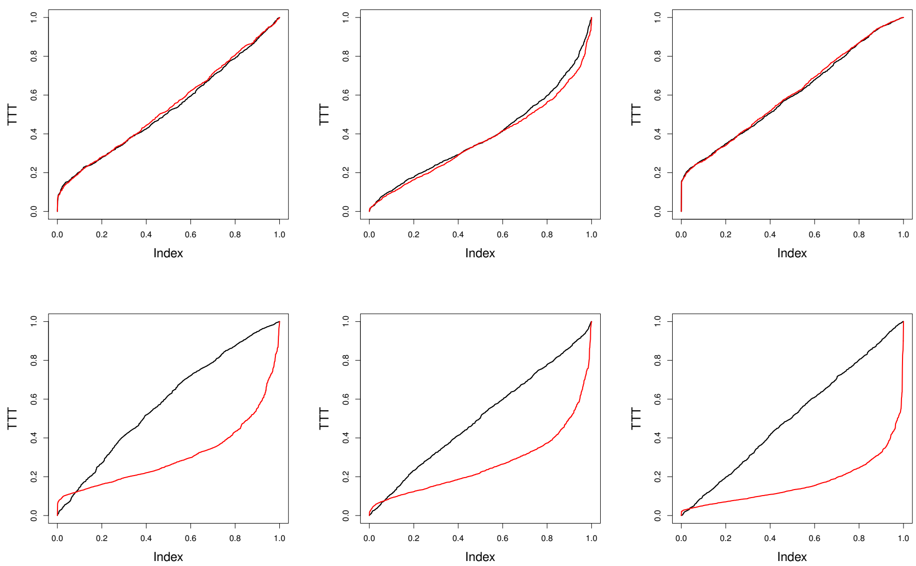

Figure 3: Scaled TTT plot for some GBS (top) and non-GBS (bottom) simulated samples and reciprocals, with $n = 10^3$.

See Figure 3 for plots of the empirical scaled TTT for simulated random samples ($n = 10^3$) for some GBS distributions (namely, a BS$(1,1)$, a BS-t_3 generated by the Student t with 3 degrees of freedom, and the GBS distribution with CDF $H_a(.;\theta)$ for $a = \theta = 0.2$, mentioned in Remark 3.1) and non-GBS distributions (half-normal, half-Student t_3 and exponential). The behavior of the statistic D_{TTT} is under investigation.

4.2. Testing for the BS model

For the case of an absolutely continuous lifetime RV T, to test the null hypothesis H_0 that the CDF of T is $F(\cdot;\theta)$, based on a random sample $(t_1, t_2, ..., t_n)$, we consider the Cramér–von Mises (CM) statistic given by

$$W_n^2 = n \int_0^{+\infty} \big(F_n^*(t) - F(t;\theta)\big)^2 \, dF(t;\theta) \,, \tag{4.3}$$

where $F_n^*(\cdot)$ is the ECDF associated to the sample. This reduces to

$$W_n^2 = \frac{1}{12\,n} + \sum_{j=1}^{n} \left(\frac{2j-1}{2n} - F(t_{j:n};\theta)\right)^2 ,$$

where $t_{1:n}, t_{2:n}, ..., t_{n:n}$ denote the corresponding order statistics. If θ is known, W_n^2 is distribution-free, in the sense that its distribution depends only on n but not on the true $F(\cdot;\theta)$, since $F(T)$ is uniformly distributed in $[0,1]$ under H_0. The asymptotic distributions were derived by Anderson and Darling [3].

As is well known, the ECDF statistics, such as W_n^2, for the case of unknown parameters usually depend on the CDF $F(\cdot;\theta)$ in H_0 as well as on n. However, in the case of a location-scale family, these statistics depend only on the family itself and n but not on the true values of the location and scale parameters, as long as an appropriate estimation method is provided (David and Johnson [14]). In some cases of a shape parameter, such as in the Gamma family, the dependence of the asymptotic and finite sample ECDF statistics on the shape parameter is slight, and tables of asymptotic percentage points were provided for different values of the parameter, to be used with the estimated values; see Stephens [41]. Another way to overcome this problem with shape parameters is to use the half-sample method introduced by Durbin [18]. This method uses a randomly chosen half of the original sample to compute the parameter estimates, say θ^*, by asymptotically efficient methods, such as maximum likelihood (ML). Then the ECDF statistics are computed with $F(\cdot;\theta^*)$ using the whole sample. The remarkable result is that asymptotically these ECDF statistics will behave like the ones for the case of known parameters. However, besides the dependence of the test conclusion on the choice of the half-sample, a considerable loss in power has been reported, namely in the case of testing for a normal or exponential distribution (see Stephens, [40] and [41]).

For a random sample $T_1, T_2, ..., T_n$ from $T \sim \text{BS}(\alpha, \beta)$, let $\theta = (\alpha, \beta)$ and $\widehat{\theta}$ and $\widetilde{\theta}$ denote respectively the ML and MM estimators of θ, and θ^* denote the ML estimator based on a randomly chosen half-sample. We shall carry out a study of the asymptotic distribution of W_n^2 in (4.3) for the case of unknown θ, using these three statistics. Thus let

$$C_n^2 = n \int_0^{+\infty} \big(F_n(t) - F(t; \widehat{\theta})\big)^2 \, dF(t; \widehat{\theta}) \tag{4.4}$$

instead of (4.3), as in Darling ([13]), or alternatively

$$C_n'^2 = n \int_0^{+\infty} \big(F_n(t) - F(t; \widetilde{\theta})\big)^2 \, dF(t; \widetilde{\theta}) \tag{4.5}$$

or

$$C_n^{*2} = n \int_0^{+\infty} \big(F_n(t) - F(t; \theta^*)\big)^2 \, dF(t; \theta^*) \,. \tag{4.6}$$

Remark 4.1. For the $\text{BS}(\alpha, \beta)$ distribution, using the asymptotic distributions of $\widehat{\beta}$ and $\widetilde{\beta}$ (see Engelhardt [19] and Ng *et al.* [33], respectively), we have $\text{var}(\widetilde{\beta}) \sim \text{var}(\widehat{\beta})$ as $\alpha \to 0$, so the relative efficiency of these two estimators tends to 1 as α decreases. Moreover, quoting Birnbaum & Saunders [11] "under this condition [$\alpha < 1/2$], which we shall later empirically verify, $\widetilde{\beta}$ is virtually the ML estimator whose optimal properties are well known", we then expect to have similar asymptotic distributions (as $n \to \infty$) in (4.4) and (4.5) when using either $\widehat{\beta}$ or $\widetilde{\beta}$, at least for small values of α.

We have computed the asymptotic percentage points for C_n^2 for testing $H_0 \colon T \sim \text{BS}(\alpha, \beta)$ with unknown parameters, based on 10^5 simulations, by the method described in Stephens [41], for significance levels 0.10, 0.05 and 0.01. This was achieved, for fixed α ($\alpha = 0.05, 0.1, 0.2, ..., 1.0$), by plotting the points obtained with simulated samples of size n ($n = 30, 40, ..., 120$) against $m = 1/n$ and extrapolating to $m = 0$. Then, the values obtained for each fixed significance level were plotted against α to extrapolate to $\alpha = 0$ by means of a polynomial fit (see Table 1). Notice that these values for $\alpha \to 0$ are almost exactly the same as for the case of a normal distribution with unknown parameters (see Table 4.7 in Stephens [41]), as expected, due to the asymptotic normality of the $\text{BS}(\alpha, \beta)$ distribution as $\alpha \to 0$ (see Remark 3.3). We also report that, for the range of α values considered, the dependence of the percentage points on n ($n \geq 30$) is slight, being negligible as α decreases and as the significance level increases. Table 1 is to be used with estimated α from the data, as mentioned before. In general, the well known data that have been fitted to a BS model have $\widehat{\alpha} < 1$, for example the lifetime data sets psi31, psi26 and psi21 in Birnbaum & Saunders [11] or the survival data set in Kundu *et al.* [27].

Table 1: Asymptotic upper-tail percentage points for C_n^2 for testing $H_0 : T \sim \mathrm{BS}(\alpha, \beta)$, both parameters unknown, based on 10^5 simulations.

α	Significance level		
	0.10	0.05	0.01
1.0	0.136	0.170	0.256
0.9	0.130	0.163	0.242
0.8	0.125	0.155	0.228
0.7	0.120	0.147	0.214
0.6	0.115	0.142	0.206
0.5	0.111	0.136	0.197
0.4	0.109	0.133	0.190
0.3	0.106	0.129	0.185
0.2	0.105	0.127	0.181
0.1	0.104	0.127	0.179
0.05	0.103	0.126	0.178
$\alpha \to 0$	0.103	0.126	0.179

We then repeated this procedure using MM instead of ML estimates of both parameters, and obtained the asymptotic percentage points for $C_n'^2$. The results, shown in Table 2, are similar to the former ones for small α values and the similarity is stronger as α decreases to 0, as expected (see Remark 4.1).

Table 2: Asymptotic upper-tail percentage points for $C_n'^2$ for testing $H_0 : T \sim \mathrm{BS}(\alpha, \beta)$, both parameters unknown, based on 10^5 simulations.

α	Significance level		
	0.10	0.05	0.01
1.0	0.126	0.156	0.230
0.9	0.123	0.151	0.222
0.8	0.120	0.148	0.215
0.7	0.117	0.144	0.208
0.6	0.113	0.139	0.200
0.5	0.111	0.136	0.194
0.4	0.109	0.133	0.190
0.3	0.106	0.130	0.186
0.2	0.105	0.127	0.181
0.1	0.103	0.126	0.179
0.05	0.103	0.126	0.179
$\alpha \to 0$	0.102	0.125	0.177

In the case of the GBS family, the percentage points for C_n^2 strongly depend on the true shape parameter α for a fixed generator X. However, if the parameters α and β are estimated by ML via the split-sample method (Durbin [18]; see also Stephens [41]), then similar results to the ones reported for testing normality and exponentiality based on CM statistic (see Stephens [40], Tables 1 and 2) were obtained. We illustrate this feature for the BS case with Table 3. This table shows the percentage points for C_n^{*2} for $\alpha = 0.1, 0.5, 1.0, 2.0, 3.0$ and $n = 20, 50, 100$, each one computed from 10^5 simulated samples at significance levels 0.10, 0.05 and 0.01, for unknown parameters estimated by the split-sample method.

Table 3: Upper-tail percentage points for C_n^{*2} for testing $H_0\colon T \sim \mathrm{BS}(\alpha, \beta)$, both parameters unknown, and upper-tail asymptotic percentage points for W_n^2 for testing $H_0\colon T \sim \mathrm{BS}(\alpha, \beta)$, both parameters known.

C_n^{*2}	n	α	Significance level		
			0.10	0.05	0.01
	20	0.1	0.373	0.490	0.755
		0.5	0.374	0.490	0.768
		1.0	0.376	0.495	0.776
		2.0	0.383	0.506	0.791
		3.0	0.372	0.491	0.778
	50	0.1	0.357	0.476	0.759
		0.5	0.355	0.472	0.745
		1.0	0.359	0.479	0.770
		2.0	0.361	0.479	0.770
		3.0	0.357	0.469	0.756
	100	0.1	0.353	0.471	0.755
		0.5	0.353	0.466	0.753
		1.0	0.354	0.473	0.761
		2.0	0.354	0.469	0.749
		3.0	0.351	0.462	0.751
W_n^2	∞		0.34730	0.46136	0.74346

The asymptotic percentage points for W_n^2 (Anderson and Darling [3]) are shown in the last row of the table. We observe that the dependence of upper-percentage points on α values is no longer strong, that it decreases as n increases and that the upper-percentage points are fairly close to the asymptotic ones for W_n^2.

We realize that a drawback of these methods is the dependence of the critical points on the unknown parameter and that there are other possible goodness-of-fit tests that can be useful in such cases; see Barros *et al.* [8] and Castro-Kuriss *et al.* [12].

5. SOME APPLICATIONS WITH DATA

In this section we analyze three well-known data sets from different areas and apply the procedures described in the previous sections to these data.

5.1. The data sets

The three data sets under analysis are (i) the survival times of 72 guinea pigs infected with tubercle bacilli in regimen 6.6 (corresponding to 4.0×10^6 bacillary units per 0.5ml), analyzed by Kundu *et al.* [27], denoted by `survpig`, (ii) the data set of lifetimes in cycles of aluminum coupons (maximum stress per cycle 31,000 psi) analyzed by Birnbaum & Saunders [11] and other authors (e.g., Ng *et al.* [33], Sanhueza *et al.* [36] and Balakrishnan *et al.* [6]), denoted by `psi31` and (iii) the data set of daily ozone concentrations collected in New York during May–September 1973, analyzed by Ferreira *et al.* [20], denoted by `ozone`. The sample dimensions are respectively $n = 72$, $n = 101$ and $n = 116$.

5.2. An introductory example

The estimation procedure based on the KS-type distance D_{KS} described in Section 4 is illustrated here by means of the data set `survpig`. For these data, all β values in the interval $[72.35, 72.92]$ minimize D_{KS}, so we took the center of this interval as its estimate, say $\beta_{KS} = 72.635$. This corresponds to a distance $D_{KS} = 6/72 = 0.0833$. See Figure 4.

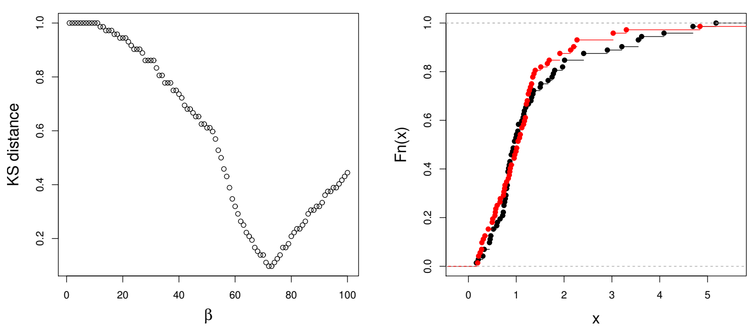

Figure 4: D_{KS} (KS-type distance) as a function of β (left) for `survpig` data and ECDF for the β-scaled sample and its reciprocal, with β estimated by minimizing D_{KS} (right).

5.3. Analyzing the data

For each of the samples, say $t = (t_1, t_2, ..., t_n)$, we applied the procedures described in the previous sections. The results are summarized in Table 4. See also the scaled TTT plots for t and t^{-1} (Figure 5) and the QQ-plots for $x = t/\widehat{\beta}$ and $y = \widehat{\beta}/t$ (Figure 6). Estimates $\widetilde{\beta}$ and β_{KS} (for β_{KS} we took the center of the interval of β values corresponding to a minimum distance D_{KS}, as explained before) were computed, as well as ML estimates of α and β for the parametric models BS(α, β) and BS-$t_\nu(\alpha, \beta)$, with ν estimated as in Azevedo *et al.* [5]. The CM-type statistics C_n^2, $C_n'^2$ and C_n^{*2} have also been computed and critical values for these statistics at significance level 5% are shown in parentheses. These values were obtained by interpolation, using Tables 1 and 2, in the first two cases, and from 10^5 simulated samples for each n ($n = 72, 101, 116$) and α ($\alpha = 0.76, 0.17, 0.98$), respectively. The classical test for symmetry about unknown location based on b_1 and the triples test were also applied to the transformed sample $y = log(t)$.

Table 4: Results for samples `survpig`, `psi31` and `ozone`.

		Data set		
		`survpig`	`psi31`	`ozone`
	n	72	101	116
	$\widetilde{\beta}$	77.4526	131.8193	28.4213
	β_{KS}	72.635	132.995	31.530
	D_{KS}	0.083	0.059	0.051
BS(α,β)	$\widehat{\beta}$	77.5348	131.8190	28.0234
	$\widehat{\alpha}$	0.7600	0.1704	0.9823
	$\widetilde{\alpha}$	0.7600	0.1704	0.9822
BS-$t_\nu(\alpha,\beta)$	ν	5	8	7
	$\widehat{\beta}$	75.5880	132.4297	30.9047
	$\widehat{\alpha}$	0.6085	0.1475	0.8074
	$\widetilde{\alpha}$	0.5887	0.1476	0.8301
	C_n^2	0.1874 (0.152)	0.0857 (0.127)	0.2071 (0.169)
	$C_n'^2$	0.1865 (0.145)	0.0857 (0.127)	0.1695 (0.156)
	C_n^{*2}	0.241 (0.470)	0.138 (0.472)	0.327 (0.471)
b_1 test	p-value	0.5886	0.3701	0.2258
triples test	p-value	0.170	0.691	0.388
	D_{TTT}	0.1068	0.0895	0.1989

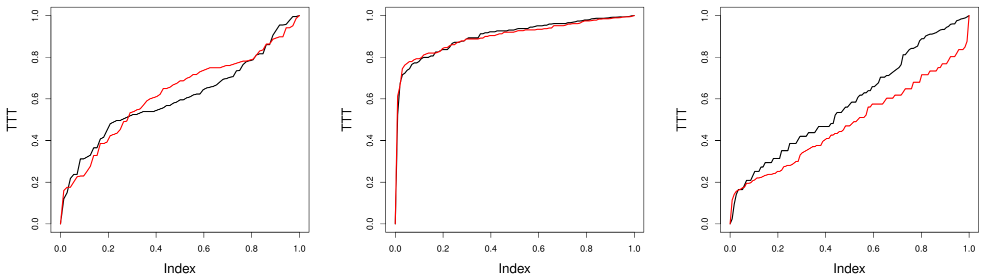

Figure 5: TTT for samples t and t^{-1} for `survpig` (left), `psi31` (center) and `ozone` (right).

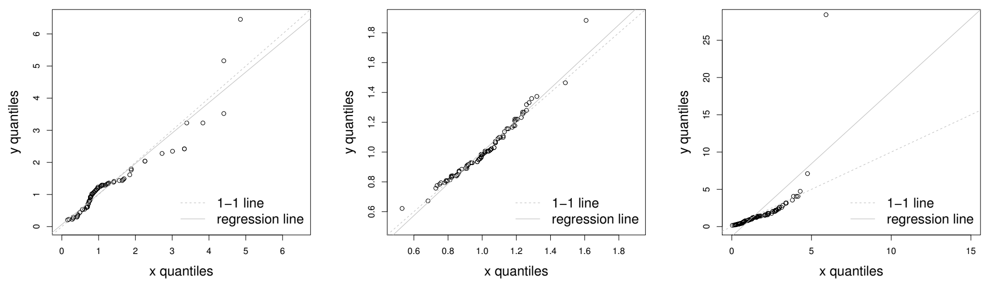

Figure 6: QQ-plots for $x = t/\widetilde{\beta}$ and $y = \widetilde{\beta}t^{-1}$, for `survpig` (left), `psi31` (center) and `ozone` (right).

The CM-type tests based on C_n^2 and $C_n'^2$ both reject the BS model for samples `survpig` and `ozone`, but not for `psi31`. The symmetry tests do not reject a GBS model for any of these samples.

Finally, the distance D_{TTT} has been computed for each sample. We also simulated the upper 5% percentage points for the distance D_{TTT} in the BS(α, β) and BS-$t_\nu(\alpha, \beta)$ models for each n (72, 101 and 116, respectively) with $\alpha = \widehat{\alpha}$ and $\beta = \widehat{\beta}$ in each case ($\nu = 5$, 8 and 7, respectively) with 10^4 simulations (see Table 5). This rules out these two particular models for `ozone`. The graphical analysis (Figures 5 and 6) also indicates that a GBS model seems reasonable for `survpig`, excellent for `psi31` and not adequate for `ozone`.

Table 5: Simulated upper-tail percentage points (at significance level 5%) for D_{TTT} assuming $T \sim$ BS$(\widehat{\alpha}, \widehat{\beta})$ or $T \sim$ BS-$t_\nu(\widehat{\alpha}, \widehat{\beta})$ ($\widehat{\alpha}$ and $\widehat{\beta}$ estimated from the three samples).

	sample		
	`survpig`	`psi31`	`ozone`
BS	0.156	0.122	0.157
BS-t_ν	0.218	0.197	0.184
	($\nu = 5$)	($\nu = 8$)	($\nu = 7$)

On the other side, as the classical symmetry b_1 test does not reject a GBS model for `ozone`, we have carried out a brief simulation study on the power of this test against several alternatives, for $n = 116$, including the extreme value Birnbaum–Saunders model generated by the Gumbel distribution for minima, denoted by $\text{EVBS}^*(\alpha, \beta, 0)$. This model was proposed by Ferreira *et al.* [20] as the best among several other models, including the BS one. The power of the test, based on 10^5 simulations, was estimated as 0.762 supposing the true model is $\text{EVBS}^*(\widehat{\alpha}, \widehat{\beta}, 0)$. If the true α in this model lies in the interval [0.6,1.0], the power decreases from 0.811 to 0.659, and can be as low as 0.044 for $\alpha = \exp(1)$. Finally, the simulated 5% upper percentage point for D_{TTT} with this model, 0.3142, also sustains the EVBS^* fit since the observed value for `ozone` is much lower (see Table 4).

6. CONCLUDING REMARKS

In this paper we derived a characterization of the GBS class related to the reciprocal property and analyzed some of its consequences. We discussed some graphical procedures to assess the fit of the GBS model to observed data, we tabulated the asymptotic percentage points for a test of the null hypothesis that the data come from a BS distribution with unknown parameters, and finally we applied the results to three well-known data sets. The case of tests for other GBS distributions, such as the ones generated by the Student t_ν or logistic distributions, is under investigation.

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