REVSTAT – Statistical Journal
Volume 15, Number 3, July 2017, 395–423

SOME INADMISSIBILITY RESULTS FOR ESTIMATING QUANTILE VECTOR OF SEVERAL EXPONENTIAL POPULATIONS WITH A COMMON LOCATION PARAMETER

Authors: Manas Ranjan Tripathy
– Department of Mathematics, National Institute of Technology Rourkela, Rourkela-769008, India
manas@nitrkl.ac.in

Somesh Kumar
– Department of Mathematics, Indian Institute of Technology Kharagpur, Kharagpur-721302, India
smsh@maths.iitkgp.ernet.in

Received: July 2014 Revised: February 2015 Accepted: September 2015

Abstract:

- Suppose independent random samples are taken from $k\,(\geq 2)$ exponential populations with a common and unknown location parameter "μ" and possibly different unknown scale parameters $\sigma_1, \sigma_2, ..., \sigma_k$ respectively. The estimation of $\underline{\theta} = (\theta_1, \theta_2, ..., \theta_k)$; where θ_i is the quantile of the i^{th} population, has been considered with respect to either a sum of squared error loss functions or sum of quadratic losses. Estimators based on maximum likelihood estimators (MLEs) and uniformly minimum variance unbiased estimators (UMVUEs) for each component θ_i have been obtained. An admissible class of estimators has been obtained. Improvement over an estimator based on UMVUEs is obtained by an application of the Brewster–Zidek technique. Further, classes of equivariant estimators are derived under affine and location groups of transformations and some inadmissibility results are proved. Finally, a numerical comparison of risk performance of all proposed estimators has been done and the recommendations are made for the use of these estimators.

Key-Words:

- *common location; complete class; equivariant estimators; inadmissibility; MLE; quantile estimation; risk comparison; simultaneous estimation; UMVUE.*

AMS Subject Classification:

- 62F10, 62C15, 62C07.

1. INTRODUCTION

Let $(X_{i1}, X_{i2}, ..., X_{in_i})$; $i = 1,2$ be independent random samples taken from two exponential populations with a common unknown location parameter μ and possibly different scale parameters σ_1, σ_2 respectively. The probability density function of the random variable X_{ij} is given by

$$f(x_{ij}\,|\,\mu,\sigma_i) = \frac{1}{\sigma_i}\exp\Big\{-\Big(\frac{x_{ij}-\mu}{\sigma_i}\Big)\Big\}, \qquad x_{ij} > \mu, \;\; -\infty < \mu < \infty, \;\; \sigma_i > 0\,; \\ j = 1,2,...,n_i\,; \;\; i = 1,2\,.$$

The p^{th} quantile of the i^{th} population is denoted by $\theta_i = \mu + \eta\sigma_i$, where $\eta = -\log(1-p) > 0$; $0 < p < 1$. We are interested in estimating the quantile vector $\underline{\theta} = (\theta_1, \theta_2)$. The loss function is taken to be either the sum of the squared errors

$$L_1(\underline{\alpha}, \underline{d}) \;=\; \sum_{i=1}^{2}(d_i - \theta_i)^2 \tag{1.1}$$

or, the sum of the quadratic losses

$$L_2(\underline{\alpha}, \underline{d}) \;=\; \sum_{i=1}^{2}\left(\frac{d_i - \theta_i}{\sigma_i}\right)^2, \tag{1.2}$$

where $\underline{\alpha} = (\mu, \sigma_1, \sigma_2)$ and $\underline{d} = (d_1, d_2)$ be an estimate of $\underline{\theta}$.

When parameters of same nature are thought to be equal, it is then customary to pool samples for inference purposes on that common parameter. This is also known as meta-analysis, and has received considerable attention from the researchers lately. For example, the problem of estimation of a common mean of two or more normal populations has been extensively studied by several authors in the recent past. The problem is popularly known as common mean problem and arises in the study of recovery of inter-block information in balanced incomplete block designs (BIBDs). For a complete bibliography and some recent results on estimation of a common mean of several normal populations one may refer to Pal and Sinha [15], Kumar [10], Mitra and Sinha [12], Pal *et al.* [13] and Tripathy and Kumar [20] and the references cited therein.

The problem of estimating a common location parameter μ of several exponential populations when the scale parameters are unknown has been studied by several authors in the recent past. The parameter μ is also referred to as the "minimum guarantee time" in the study of reliability. This problem was probably first considered by Ghosh and Razmpour [5]. They have obtained the maximum likelihood estimator (MLE), a modified maximum likelihood estimator (MMLE) and the uniformly minimum variance unbiased estimator (UMVUE). They have also compared numerically the risk values of all these estimators with respect to the squared error loss function whereas the MLE and the MMLE have been compared asymptotically in terms of their bias and mean squared errors (MSEs). Pal and Sinha [14] considered this problem from a decision theoretic point of view. They proposed a class of improved estimators which are better than the MLE in terms of MSE as well as Pitman measure of closeness (PMC). However, these improved estimators are different from the MMLE and the UMVUE. Jin and Pal [9] obtained a wide class of estimators which dominate the MLE under a class of convex loss functions. Jin and Crouse [7] proposed a larger class of estimators for μ which includes the MMLE and the UMVUE for special choices of their constants (see Equation (3.1) in [7]). They obtained estimators which dominate the MLE using a class of convex loss functions.

For this particular model, the problem of estimation of quantiles is important and also interesting for its practical applications. Quantiles of exponential populations are very much useful in the study of reliability, life testing, and survival analysis. For some applications of quantiles of exponential populations we refer to Epstein and Sobel [3] and Saleh [18]. Estimation of quantiles $\theta_1 = \mu + \eta\sigma_1$, of an exponential population was probably first considered by Rukhin and Strawderman [17] using a decision theoretic approach. They proved that the best affine equivariant estimator (BAEE) for the quantile θ_1 is inadmissible when either $0 \leq \eta < \frac{1}{n}$ or $\eta > 1 + \frac{1}{n}$ where $n \geq 2$ is the sample size. Rukhin [16] proved its admissibility when $\frac{1}{n} \leq \eta \leq 1 + \frac{1}{n}$. He also obtained a class of minimax estimators for $\eta > 1 + \frac{1}{n}$. This class contains some generalized Bayes estimators. One of these generalized Bayes estimators is shown to be admissible within a class of scale equivariant estimators.

For the model studied in this paper, Sharma and Kumar [19] and Kumar and Sharma [11] considered estimation of the quantiles $\theta_1 = \mu + \eta\sigma_1$ of the first population when the other $k-1$ $(k \geq 2)$ populations are available. They show that the MLE, the UMVUE and the BAEE based on the first sample alone can be improved by using other $k-1$ samples. They have also obtained a general inadmissibility result for the class of affine equivariant estimators for $0 \leq \eta < \frac{1}{n}$. Jin and Crouse [8] considered the problem of estimating the quantile $\theta_i = \mu + \eta\sigma_i$ of the i^{th} population. They established an identity for the exponential distributions, and using it, compared the risk functions of the UMVUE and the MLE. They also proposed a class of estimators which dominate the MLE and the UMVUE.

It is interesting to note that all the above work relates to estimating either the common location parameter μ or a component θ_i of the vector $\underline{\theta}$ of quantiles. From a theoretical as well as a practical viewpoint, it is important to consider the problem of simultaneous estimation of $\underline{\theta}$. For example, suppose an electronic item is produced by several manufacturers and lifetimes of these follow exponential distributions. It is very likely that the average lives of items from different manufacturers will be different due to quality specifications used by them.

However, due to competition in the market, they will maintain a common minimum guarantee time. Then the problem of simultaneous estimation of average lives is a special case of the problem of simultaneous estimation of the vector of quantiles. One may refer to Ghosh and Auer [4], Berger [1] and Gupta [6] for some results on the simultaneous estimation of parameters.

In this paper, we consider the general problem of estimating the vector of quantiles of several exponential populations with a common location but different scale parameters. In Section 2, some basic estimators of the quantile vector are proposed based on the MLE, the MMLE and the UMVUE of each component. In Section 3, we consider classes of affine and location equivariant estimators and prove some inadmissibility results. In Section 4, we extend some of these results to $k\,(\geq 2)$ exponential populations. A detailed numerical comparison of risk values for several proposed estimators has been done by using Monte-Carlo simulations in Section 5. Also recommendations are made for using these estimators. Certain proofs have been given in the Appendix.

2. SOME BASIC RESULTS & IMPROVEMENT OVER UMVUE

In this section we derive some baseline estimators for the quantile vector $\underline{\theta}$ and obtain an estimator which dominates the UMVUE using a result of Brewster and Zidek (Brewster and Zidek [2]).

2.1. Some Basic Estimators

Suppose $(X_{i1}, X_{i2}, ..., X_{in_i})$; $i=1,2$ are independent random samples taken from two exponential populations $Ex(\mu,\sigma_1)$ and $Ex(\mu,\sigma_2)$ having the probability density functions,

$$f(x_{ij}) = \frac{1}{\sigma_i}\,\exp\Big\{-\Big(\frac{x_{ij}-\mu}{\sigma_i}\Big)\Big\}\,, \qquad x_{ij}>\mu\,,\;\; -\infty<\mu<\infty\,,\;\; \sigma_i>0\,, \qquad j=1,2,...,n_i\,;\;\; i=1,2\,,$$

respectively. We are interested in estimating the quantile vector $\underline{\theta}=(\theta_1,\theta_2)$, where $\theta_i=\mu+\eta\sigma_i$ denotes the quantile of the i^{th} population, $i=1,2$. The loss function is taken to be either the sum of the squared errors (1.1) or the sum of the quadratic losses (1.2).

Let us denote $X_i=\min(X_{i1},X_{i2},...,X_{in_i})$ and $Y_i=\frac{1}{n_i}\sum_{j=1}^{n_i}X_{ij}$; $i=1,2$. Further define $Z=\min(X_1,X_2)$, $T_1=Y_1-Z$, and $T_2=Y_2-Z$. Here Y_1 and Y_2 are the means of the first and the second samples respectively. Then (Z,T_1,T_2)

is a complete sufficient statistic. The random variables Z and $\underline{T} = (T_1, T_2)$ are independently distributed. The probability density function of Z is given by

$$f_Z(z) = a \exp\bigl(-a(z-\mu)\bigr)\,, \qquad z > \mu\,, \quad -\infty < \mu < \infty\,,$$

where $a = \frac{n_1}{\sigma_1} + \frac{n_2}{\sigma_2}$. The joint probability density function of T_1 and T_2 can be obtained from Ghosh and Razmpour [5] by using a simple transformation, and is given by

$$f_{\underline{T}}(\underline{t}) \;=\; \frac{n_1^{n_1} n_2^{n_2}}{\sigma_1^{n_1} \sigma_2^{n_2} a} \left[\frac{t_1^{n_1-1} t_2^{n_2-2}}{\Gamma n_1\, \Gamma(n_2-1)} + \frac{t_1^{n_1-2} t_2^{n_2-1}}{\Gamma n_2\, \Gamma(n_1-1)}\right] \exp(-n_1 t_1/\sigma_1 - n_2 t_2/\sigma_2)\,, \qquad t_1, t_2 > 0\,.$$

The MLE of μ and σ_i are $\hat{\mu} = Z$, and $\hat{\sigma}_i = T_i$, $i = 1, 2$ respectively. Thus collecting the MLEs for each component we obtain the estimator for quantile vector $\underline{\theta}$ as

$$\underline{\delta}_{ML} \;=\; \bigl(Z + \eta T_1, Z + \eta T_2\bigr)\,,$$

and we call it the MLE of $\underline{\theta}$. Further noticing $E(Z) = \mu + a^{-1}$, the MLE $\underline{\delta}_{ML}$ of $\underline{\theta}$ can be modified and we call this a modified MLE for the vector $\underline{\theta}$ and is given by

$$\underline{\delta}_{MM} \;=\; \bigl(Z - \hat{a}^{-1} + \eta T_1,\, Z - \hat{a}^{-1} + \eta T_2\bigr)\,,$$

where $\hat{a} = \frac{n_1}{T_1} + \frac{n_2}{T_2}$.

Next we collect the UMVUEs of θ_i for each component and form an estimator for the quantile vector $\underline{\theta}$. It is easy to see that $E(T_j) = \sigma_j - a^{-1}$; $j = 1, 2$ and $E\bigl[\bigl(\sum_{i=1}^{2}(n_i - 1)\, T_i^{-1}\bigr)^{-1}\bigr] = a^{-1}$. Using these results and the fact that $(Z, \underline{T})$ is a complete sufficient statistic, we get the uniformly minimum variance unbiased estimator for each component θ_i as $Z + (\eta-1)T^* + \eta T_i$, where we denote $T^* = \bigl(\sum_{i=1}^{2}(n_i-1)\, T_i^{-1}\bigr)^{-1}$. Now collecting the UMVUEs for each component θ_i, we form an estimator for the quantile vector $\underline{\theta}$, denoted as $\underline{\delta}_{MV}$ and is given by

$$\underline{\delta}_{MV} = \Bigl(Z + \eta T_1 + (\eta-1)\, T^*,\, Z + \eta T_2 + (\eta-1)\, T^*\Bigr)\,.$$

The expressions for the risk functions of $\underline{\delta}_{ML}$, $\underline{\delta}_{MM}$ and $\underline{\delta}_{MV}$ with respect to the loss (1.2) are obtained as follows:

$$R(\underline{\delta}_{ML}, \underline{\theta}) \;=\; \eta^2 \left(\frac{1}{n_1} + \frac{1}{n_2}\right) + \frac{2\,(1-\eta)\,(1+\tau^2)}{(n_1 + n_2\,\tau)^2}\,,$$

$$\begin{aligned} R(\underline{\delta}_{MM}, \underline{\theta}) \;&=\; \eta^2 \left(\frac{1}{n_1} + \frac{1}{n_2}\right) + \frac{2\,(1-\eta)\,(1+\tau^2)}{(n_1 + n_2\,\tau)^2} \\ &\quad + \left(\frac{1}{\sigma_1^2} + \frac{1}{\sigma_2^2}\right) \left(\frac{2\,(\eta-1)}{a}\, ES + ES^2\right), \end{aligned}$$

$$R(\underline{\delta}_{MV}, \underline{\theta}) \;=\; \eta^2 \left(\frac{1}{n_1} + \frac{1}{n_2}\right) + \frac{4\,(1-\eta)\,(1+\tau^2)}{(n_1 + n_2\,\tau)^2} + 2\,(\eta-1)^2\, ET^{*2}\,,$$

where $S = 1/\hat{a}$ and $\tau = \sigma_1/\sigma_2 > 0$.

In the rest of the paper, when we say UMVUE for the quantile vector $\underline{\theta}$, we mean "the collection of the UMVUEs for each component θ_i and form a vector" to get the estimator for the quantile vector $\underline{\theta}$.

2.2. An Estimator Dominating the UMVUE

In this section, we propose an estimator for the quantile vector $\underline{\theta}$, which improves upon the UMVUE for the quantile vector $\underline{\theta}$, with respect to the loss function (1.1). Let us consider a class of estimators for the quantile vector $\underline{\theta}$ as

$$D = \left\{ \underline{\delta}_{\mathbf{c}} \colon \underline{\delta}_{\mathbf{c}} = (\delta_{c_1}, \delta_{c_2});\ c_1, c_2 \in \mathbb{R} \right\},$$

where we denote $\delta_{c_j} = Z + \eta c_j T_j + (\eta - 1) T^*$; $j = 1, 2$.

Now for the class of estimators $D = \left\{ \underline{\delta}_{\mathbf{c}} \colon \underline{\delta}_{\mathbf{c}} = (\delta_{c_1}, \delta_{c_2});\ c_1, c_2 \in \mathbb{R} \right\}$, let us define

$$\mathbf{c}^* = \Big(\min\big\{ \max(c_1, a_1), b_1 \big\},\ \min\big\{ \max(c_2, a_2), b_2 \big\} \Big) \tag{2.1}$$

and

$$\mathbf{c}_* = \Big(\min\big\{ \max(c_1, c_1^+), d_1 \big\},\ \min\big\{ \max(c_2, c_2^+), d_2 \big\} \Big) \tag{2.2}$$

where $a_j = \frac{\eta n_j (n_j - 2) + 1}{\eta n_j (n_j - 1)}$, $b_j = \frac{n_j}{n_j + 1}$, $d_j = \max\{a_j, b_j\}$, $c_j^+ = \hat{c}_j(\lambda_j^+)$, and $\lambda_j^+ = \big\{ (n_j + 1) - \sqrt{(n_j + 1)^2 - 4\eta n_j} \big\} / 2n_j$; $j = 1, 2$. Next we have the following inadmissibility result for estimators in the class D.

Theorem 2.1. *Let D be the class of estimators for the quantile vector $\underline{\theta}$, and define the functions $\mathbf{c}^*$ and $\mathbf{c}_*$ as in (2.1) and (2.2) respectively. Let the loss function be* (1.1).

- **(i)** *The estimator $\underline{\delta}_{\mathbf{c}}$ is inadmissible and is improved by $\underline{\delta}_{\mathbf{c}^*}$ if $\mathbf{c} \neq \mathbf{c}^*$, when $\eta \geq 1$.*
- **(ii)** *The estimator $\underline{\delta}_{\mathbf{c}}$ is inadmissible and is improved by $\underline{\delta}_{\mathbf{c}_*}$ if $\mathbf{c} \neq \mathbf{c}_*$ when $0 < \eta < 1$.*

Proof: See Appendix.

Corollary 2.1.

- **(i)** *Let $\eta \geq 1$. The class of estimators $\big\{ \underline{\delta}_{\mathbf{c}} = (\delta_{c_1}, \delta_{c_2}) \colon a_j \leq c_j \leq b_j;\ j = 1, 2 \big\}$ is essentially complete in D.*
- **(ii)** *Let $0 < \eta < 1$. The class of estimators $\big\{ \underline{\delta}_{\mathbf{c}} = (\delta_{c_1}, \delta_{c_2}) \colon c_j^+ \leq c_j \leq d_j;\ j = 1, 2 \big\}$ is essentially complete in D.*

The class of estimators D also contains the UMVUE for the quantile vector $\underline{\theta}$ when $c_1 = c_2 = 1$. Consequently, the UMVUE $\underline{\delta}_{MV}$ is inadmissible. The result we write as a theorem which is immediate. Let $p_1 = \min\{\frac{1}{n_1}, \frac{1}{n_2}\}$, $p_2 = \max\{\frac{1}{n_1}, \frac{1}{n_2}\}$, $q_1 = \min\{\frac{n_1+1}{2n_1}, \frac{n_2+1}{2n_2}\}$ and $q_2 = \max\{\frac{n_1+1}{2n_1}, \frac{n_2+1}{2n_2}\}$.

Theorem 2.2. *Let the loss function be (1.1).*

- **(i)** *If $\eta \geq 1$, then the uniformly minimum variance unbiased estimator $\underline{\delta}_{MV}$ for the quantile vector $\underline{\theta}$ is inadmissible and is improved by the estimator $\underline{\delta}_{IMV} = (\delta_{b_1}, \delta_{b_2})$. Further the class $\{\underline{\delta}_{\mathbf{c}} = (\delta_{c_1}, \delta_{c_2})\colon a_j \leq c_j \leq b_j;\ j = 1, 2\}$ is essentially complete in D.*
- **(ii)** *If $q_2 \leq \eta < 1$, then the estimator $\underline{\delta}_{MV}$ is inadmissible and is improved by $\underline{\delta}_{IMV} = (\delta_{b_1}, \delta_{b_2})$. The class of estimators $\{\underline{\delta}_{\mathbf{c}} = (\delta_{c_1}, \delta_{c_2})\colon c_j^+ \leq c_j \leq b_j;\ j = 1, 2\}$ is essentially complete in D.*
- **(iii)** *If $p_2 \leq \eta \leq q_1$, then the estimator $\underline{\delta}_{MV}$ is inadmissible and is improved by $\underline{\delta}_{IMV} = (\delta_{a_1}, \delta_{a_2})$. The class of estimators $\{\underline{\delta}_{\mathbf{c}} = (\delta_{c_1}, \delta_{c_2})\colon c_j^+ \leq c_j \leq a_j;\ j = 1, 2\}$ is essentially complete in D.*
- **(iv)** *If $0 \leq \eta < p_1$, then the estimator $\underline{\delta}_{MV}$ is admissible in the class D. The class of estimators $\{\underline{\delta}_{\mathbf{c}} = (\delta_{c_1}, \delta_{c_2})\colon c_j^+ \leq c_j \leq a_j;\ j = 1, 2\}$ is essentially complete in D.*
- **(v)** *Let $p_1 < \eta < p_2$. If $\frac{1}{n_1} < \eta < \frac{1}{n_2}$, then the estimator $\underline{\delta}_{MV}$ is inadmissible and is improved by either $(\delta_{a_1}, \delta_{c_2})$ (when $\eta < \frac{n_1+1}{2n_1}$) or $(\delta_{b_1}, \delta_{c_2})$ (when $\eta \geq \frac{n_1+1}{2n_1}$) where $c_2 = 1$. If $\frac{1}{n_2} < \eta < \frac{1}{n_1}$, then the estimator $\underline{\delta}_{MV}$ is inadmissible and is improved by either $(\delta_{c_1}, \delta_{a_2})$ (when $\eta < \frac{n_2+1}{2n_2}$) or $(\delta_{c_1}, \delta_{b_2})$ (when $\eta \geq \frac{n_2+1}{2n_2}$) where $c_1 = 1$.*
- **(vi)** *Let $q_1 < \eta < q_2$. If $\frac{n_1+1}{2n_1} < \eta < \frac{n_2+1}{2n_2}$, then the estimator $\underline{\delta}_{MV}$ is inadmissible and is improved by either $(\delta_{b_1}, \delta_{a_2})$ (when $\eta > \frac{1}{n_2}$) or $(\delta_{b_1}, \delta_{c_2})$ (when $\eta < \frac{1}{n_2}$) where $c_2 = 1$. If $\frac{n_2+1}{2n_2} < \eta < \frac{n_1+1}{2n_1}$, then the estimator $\underline{\delta}_{MV}$ is inadmissible and is improved by either $(\delta_{a_1}, \delta_{b_2})$ (when $\eta > \frac{1}{n_1}$) or $(\delta_{c_1}, \delta_{b_2})$ (when $\eta < \frac{1}{n_1}$) where $c_1 = 1$.*

Proof: The proof is immediate as an application of Theorem 2.1. □

Applying the above Theorem 2.2 it is easy to write the estimator which improves upon $\underline{\delta}_{MV}$. However, we give the expression only for the case $\eta \geq q_2$ and $p_2 < \eta \leq q_1$ below. The expressions for other cases can be written easily:

$$\underline{\delta}_{IMV} = \begin{cases} \Big(Z + \eta b_1 T_1 + (\eta - 1) T^*,\ Z + \eta b_2 T_2 + (\eta - 1) T^*\Big), & \text{if } \eta \geq q_2\,, \\ \Big(Z + \eta a_1 T_1 + (\eta - 1) T^*,\ Z + \eta a_2 T_2 + (\eta - 1) T^*\Big), & \text{if } p_2 < \eta \leq q_1\,. \end{cases}$$

3. INADMISSIBILITY OF EQUIVARIANT ESTIMATORS FOR QUANTILES

In this section, we consider affine and location class of equivariant estimators for the quantile vector $\underline{\theta}$. We derive sufficient conditions for improving estimators in these classes and as a consequence we prove some complete class results.

3.1. Affine Equivariant Estimators

Let us consider the affine group of transformations, $G_A = \{g_{a,b}\colon g_{a,b}(x) = ax+b,\ a>0,\ b\in\mathbb{R}\}$. Under the transformation $g_{a,b}$, we have $X_{ij} \to aX_{ij}+b$, $Z \to aZ+b$, $T_i \to aT_i$, $\sigma_i \to a\sigma_i$, $\mu \to a\mu+b$, and $\theta_i = \mu+\eta\sigma_i \to a\theta_i+b$; $i=1,2$. So $\underline{\theta} = (\theta_1,\theta_2) \to a\underline{\theta}+b\underline{e}$, where $\underline{e}=(1,1)$. The estimation problem is invariant if we take the loss function as the sum of the affine invariant loss functions (1.2). The invariance loss condition is

$$\begin{aligned} L\big(\bar{g}_{a,b}(\underline{\alpha}),\tilde{d}\big) &= \sum_{i=1}^{2}\left(\frac{a\theta_i+b-\tilde{d}_i}{a\sigma_i}\right)^2 \\ &= L(\underline{\alpha},\underline{d})\,, \end{aligned}$$

which is satisfied if $\tilde{d}_i = ad_i+b = \tilde{g}_{a,b}(d_i)$, $i=1,2$. Here $\underline{\alpha} = (\mu,\sigma_1,\sigma_2)$. Therefore an affine equivariant estimator satisfies

$$\underline{\delta}(aZ+b, aT_1, aT_2) \;=\; a\underline{\delta}(Z,T_1,T_2)+b\underline{e}\,.$$

Substituting $b=-aZ$ where $a=1/T_1$, we get

$$\underline{\delta}\left(0,1,\frac{T_2}{T_1}\right) \;=\; \frac{1}{T_1}\left[\underline{\delta}(Z,T_1,T_2)-Z\underline{e}\right]\,.$$

From the above relation, we get the form of an affine equivariant estimator as

$$\begin{aligned} \underline{\delta}(Z,T_1,T_2) &= Z\underline{e}+T_1\underline{\Psi}(W) \\ &= \underline{\delta}_{\underline{\Psi}}\,, \qquad \text{say}\,, \end{aligned} \tag{3.1}$$

where $W=\frac{T_2}{T_1}$. To proceed further we denote $\eta_1=\frac{\eta n_1-1}{n_1+n_2}$, and $\eta_2=\frac{n_2 w}{n_1+n_2}\big(\eta-\frac{1}{n_2}\big)$.

Let us define the following functions:

$$\Psi_1^* = \begin{cases} \eta_1\,, & \text{if } 0\le w\le \frac{1}{1-\eta n_1}\,, \\ \hat{\Psi}_1(\tau^+,w)\,, & \text{if } w>\frac{1}{1-\eta n_1}\,, \end{cases} \tag{3.2}$$

where $\tau^+ = -\frac{n_1}{n_2} + \frac{1}{n_2}\sqrt{\frac{n_1(w-1)}{\eta w}}$, and

$$\Psi_2^* = \begin{cases} \eta_2\,, & \text{if } w \geq 1-\eta n_2\,, \\ \hat{\Psi}_2(\alpha^+, w)\,, & \text{if } w < 1-\eta n_2\,, \end{cases} \tag{3.3}$$

where $\alpha^+ = \frac{\eta n_1}{1-\eta n_2 - w} + \frac{n_1\sqrt{\eta n_2(1-w)}}{n_2(1-\eta n_2-w)}$.

Next, for the affine equivariant estimator $\underline{\delta}_{\underline{\Psi}}$ define functions $\underline{\Psi}_0$, $\underline{\Psi}^0$, $\underline{\Psi}_{11}$ and $\underline{\Psi}_{22}$ as follows:

$$\underline{\Psi}_0 = \big(\max(\Psi_1, \Psi_1^*),\, \max(\Psi_2, \Psi_2^*)\big)\,, \tag{3.4}$$

$$\underline{\Psi}^0 = \big(\max(\Psi_1, \eta_1),\, \max(\Psi_2, \eta_2)\big)\,, \tag{3.5}$$

$$\underline{\Psi}_{11} = \big(\max(\Psi_1, \eta_1),\, \max(\Psi_2, \Psi_2^*)\big)\,, \tag{3.6}$$

$$\underline{\Psi}_{22} = \big(\max(\Psi_1, \Psi_1^*),\, \max(\Psi_2, \eta_2)\big)\,. \tag{3.7}$$

Let p_1 and p_2 be defined as in Section 2. Next we prove an inadmissibility result for estimators which are equivariant under the affine group of transformations.

Theorem 3.1. *Let the loss function be (1.2) and the functions* $\underline{\Psi}_0$, $\underline{\Psi}^0$, $\underline{\Psi}_{11}$ *and* $\underline{\Psi}_{11}$ *be defined as in (3.4), (3.5), (3.6) and (3.7) respectively.*

- **(i)** *The estimator* $\underline{\delta}_{\underline{\Psi}}$ *is inadmissible and is improved by* $\underline{\delta}_{\underline{\Psi}_0}$ *if there exist some values of parameters* $\underline{\alpha}$ *such that* $P(\underline{\delta}_{\underline{\Psi}} \neq \underline{\delta}_{\underline{\Psi}_0}) > 0$ *when* $0 < \eta < p_1$.
- **(ii)** *The estimator* $\underline{\delta}_{\underline{\Psi}}$ *is inadmissible and is improved by* $\underline{\delta}_{\underline{\Psi}^0}$ *if there exist some values of parameters* $\underline{\alpha}$ *such that* $P(\underline{\delta}_{\underline{\Psi}} \neq \underline{\delta}_{\underline{\Psi}^0}) > 0$ *when* $\eta \geq p_2$.
- **(iii)** *Let* $p_1 \leq \eta < p_2$. *If* $\frac{1}{n_1} \leq \frac{1}{n_2}$, *then the estimator* $\underline{\delta}_{\underline{\Psi}}$ *is inadmissible and is improved by* $\underline{\delta}_{\underline{\Psi}_{11}}$ *if there exist some values of parameters* $\underline{\alpha}$ *such that,* $P(\underline{\delta}_{\underline{\Psi}} \neq \underline{\delta}_{\underline{\Psi}_{11}}) > 0$. *If* $\frac{1}{n_2} \leq \frac{1}{n_1}$, *then the estimator* $\underline{\delta}_{\underline{\Psi}}$ *is inadmissible and is improved by* $\underline{\delta}_{\underline{\Psi}_{22}}$ *if there exist some values of parameters* $\underline{\alpha}$ *such that,* $P(\underline{\delta}_{\underline{\Psi}} \neq \underline{\delta}_{\underline{\Psi}_{22}}) > 0$.

Proof: For proof see Appendix.

Remark 3.1. The Theorem 3.1 is basically a complete class theorem for affine equivariant estimators. It says that any affine equivariant estimator of the form (3.1) will be inadmissible if $P\{(\Psi_1 < \Psi_1^*) \cup (\Psi_2 < \Psi_2^*)\} > 0$ when $\eta < p_1$ and $P\{(\Psi_1 < \eta_1) \cup (\Psi_2 < \eta_2))\} > 0$ for $\eta \geq p_2$. A similar type of statement holds for the case $p_1 \le \eta \le p_2$. However, for small values of η and for small sample sizes the improvements over the MLE and the MMLE are very marginal and we omit the risk values in the tables. For $\eta > p_2$, improvement over these is not possible by using the result of Theorem 3.1. Improvement over $\underline{\delta}_{MV}$ has been shown in the Tables 1–3 for $0 < \eta < p_1$.

Remark 3.2. The results of the Theorem 3.1 will remain valid, if instead of the loss function (1.2), we use any sum of the weighted squared error loss functions.

3.2. Location Equivariant Estimator

Let us introduce the location group of transformations, $G_L = \{g_a\colon g_a(x) = x + a,\ a \in \mathbb{R}\}$ to our model. Under the transformation g_a, $X_{ij} \to X_{ij} + a$, $X_i \to X_i + a$, $Z \to Z + a$, $T_i \to T_i$, $\sigma_i \to \sigma_i$, $\mu \to \mu + a$, and $\underline{\theta} = (\theta_1, \theta_2) \to (\theta_1 + a, \theta_2 + a) = \underline{\theta} + a\underline{e}$, where $\theta_i = \mu + \eta\sigma_i$; $i = 1, 2$.

The estimation problem will be invariant if we choose the loss function as the sum of the squared error loss functions (1.1). The location equivariant estimator $\underline{\delta}$ must satisfy the relation

$$\underline{\delta}(Z + a, T_1, T_2) \;=\; a\underline{e} + \underline{\delta}(Z, T_1, T_2)\;.$$

Substituting $a = -Z$, we get

$$\underline{\delta}(0, T_1, T_2) \;=\; \underline{\delta}(Z, T_1, T_2) - Z\underline{e}\;.$$

From this relation we get the form of a location equivariant estimator as

$$\begin{aligned} \underline{\delta}(Z, T_1, T_2) \;&=\; Z\underline{e} + \underline{\psi}(T_1, T_2) \\ &=\; \underline{\delta}_{\underline{\psi}}\,, \qquad \text{say}\,, \end{aligned} \tag{3.8}$$

where $\underline{\psi}(T_1, T_2) = (\psi_1(T_1, T_2), \psi_2(T_1, T_2))$.

For the location equivariant estimator $\underline{\delta}_{\underline{\psi}} = (\delta_{\psi_1}, \delta_{\psi_2})$ let us define functions, $\underline{\psi}^0$, $\underline{\psi}_{11}$ and $\underline{\psi}_{22}$ as

$$\underline{\psi}^0 \;=\; \big(\max(0, \psi_1), \max(0, \psi_2)\big)\,, \tag{3.9}$$

$$\underline{\psi}_{11} \;=\; \big(\max(0, \psi_1), \psi_2\big)\,, \tag{3.10}$$

and

$$\underline{\psi}_{22} \;=\; \big(\psi_1, \max(0, \psi_2)\big)\,. \tag{3.11}$$

Next we prove an inadmissibility result for estimators which are invariant under the location group of transformations.

Theorem 3.2. *Let the loss function be (1.1) and the functions* $\underline{\psi}^0$, $\underline{\psi}_{11}$ *and* $\underline{\psi}_{22}$ *be defined as in (3.9), (3.10) and (3.11) respectively.*

- **(i)** *When* $\eta \geq p_2$ *the estimator* $\underline{\delta}_{\underline{\psi}}$ *is inadmissible and is improved by* $\underline{\delta}_{\underline{\psi}^0}$ *if there exist some values of the parameters* $\underline{\alpha}$ *such that* $P_{\underline{\alpha}}(\underline{\delta}_{\underline{\psi}} \neq \underline{\delta}_{\underline{\psi}^0}) > 0$.
- **(ii)** *Let* $p_1 \leq \eta < p_2$. *If* $\frac{1}{n_1} \leq \eta < \frac{1}{n_2}$, *then the estimator* $\underline{\delta}_{\underline{\psi}}$ *is inadmissible and is improved by* $\underline{\delta}_{\underline{\psi}_{11}}$ *if there exist some values of parameters* $\underline{\alpha}$ *such that* $P_{\underline{\alpha}}(\underline{\delta}_{\underline{\psi}} \neq \underline{\delta}_{\underline{\psi}_{11}}) > 0$. *If* $\frac{1}{n_2} \leq \eta < \frac{1}{n_1}$ *the estimator* $\underline{\delta}_{\underline{\psi}}$ *is inadmissible and is improved by* $\underline{\delta}_{\underline{\psi}_{22}}$ *if there exist some values of parameters* $\underline{\alpha}$ *such that* $P_{\underline{\alpha}}(\underline{\delta}_{\underline{\psi}} \neq \underline{\delta}_{\underline{\psi}_{22}}) > 0$.
- **(iii)** *For* $\eta < p_1$ *the class of estimators (3.8) is an essentially complete class. The estimator* $\underline{\delta}_{\underline{\psi}}$ *can not be improved by using Theorem 3.2.*

Proof: The proof is similar to the arguments used in proving the Theorem 3.1. □

Remark 3.3. The above Theorem 3.2 is also a complete class result. Basically it says that any location equivariant estimator for the quantile vector $\underline{\theta} = (\theta_1, \theta_2)$ of the form (3.8) is inadmissible if $P_{\underline{\alpha}}\{(\psi_1 < 0) \cup (\psi_2 < 0)\} > 0$ for $\eta \geq \max(\frac{1}{n_1}, \frac{1}{n_2})$. A similar type of statement holds for the case $p_1 \leq \eta \leq p_2$.

Remark 3.4. It can be further noticed that all the estimators considered such as $\underline{\delta}_{ML}$, $\underline{\delta}_{MM}$ and $\underline{\delta}_{MV}$ belong to the class of estimators obtained in (3.6), with choices of $\underline{\psi} = (\psi_1, \psi_2)$ as $(\eta T_1, \eta T_2)$, $\left(\eta T_1 - \frac{T_1 T_2}{n_2 T_1 + n_1 T_2}, \eta T_2 - \frac{T_1 T_2}{n_2 T_1 + n_1 T_2}\right)$ and $\left(\eta T_1 + \frac{(\eta-1) T_1 T_2}{(n_1-1) T_2 + (n_2-1) T_1}, \eta T_2 + \frac{(\eta-1) T_1 T_2}{(n_1-1) T_2 + (n_2-1) T_1}\right)$ respectively. But none of these can be improved by using the result of Theorem 3.2 as the values of ψ_1 and ψ_2 fall within the interval $[0, +\infty)$ when $\eta \geq \max\left(\frac{1}{n_1}, \frac{1}{n_2}\right)$ with probability 1. However, an example where our result will be useful is as follows: suppose we consider an estimator for $\underline{\theta}$ as $\underline{\delta} = (Z - \eta T_1, Z - \eta T_2)$ or any estimator of the form $(Z - g_1(T_1, T_2), Z - g_2(T_1, T_2))$, with $g_1(t_1, t_2) > 0$, or $g_2(t_1, t_2) > 0$ and $\eta \geq \max\left(\frac{1}{n_1}, \frac{1}{n_2}\right)$. Certainly, these estimators fall in the class (3.8) with $\psi_1 < 0$ or $\psi_2 < 0$. The improved estimator for these are obtained as $\underline{\delta}^* = (Z, Z)$.

Example 3.1. An example of a practical situation where the model of this paper is applicable is presented here. Suppose μ is the common minimum guaranteed time in years of two brands of electronics products say brand A and brand B. It is most likely that the mean residual life times (σ_1 and σ_2) will be different. On the basis of random samples of sizes 10 from brand A and B, the following summary data has been recorded. Here $Z = 7.82$, $T_1 = 12.49$ and $T_2 = 15.44$. Suppose $\eta = 3.0$, then the estimators for the quantile vector are obtained as $\underline{\delta}_{ML} = (45.31, 54.14)$, $\underline{\delta}_{MM} = (44.62, 53.45)$, $\underline{\delta}_{MV} = (46.85, 55.68)$, and $\underline{\delta}_{IMV} = (43.44, 51.47)$. In this situation, the estimator $\underline{\delta}_{IMV} = (43.44, 51.47)$ is recommended for use.

Example 3.2 (Simulated Data)**.** The following two data sets A and B of sizes each 10 and 12 has been generated from two exponential populations for illustration purpose. We have taken $\mu = 5.0$, $\sigma_1 = 5$ and $\sigma_2 = 10$. The sample values have been written up to 3 decimal places only:

A: 16.555, 9.685, 11.863, 11.248, 6.894, 20.933, 6.435, 8.573, 18.745, 9.036,

B: 5.455, 6.806, 10.667, 13.687, 11.739, 9.006, 7.612, 18.846, 23.978, 21.418, 10.639, 13.061.

Here $Z = 5.455$, $T_1 = 6.541$ and $T_2 = 7.287$. Suppose, $\eta = 0.001$, then the estimators for the quantile vector $\underline{\theta}$ are obtained as, $\underline{\delta}_{ML} = (5.462, 5.462)$, $\underline{\delta}_{MM} = (5.147, 5.147)$, $\underline{\delta}_{MV} = (5.115, 5.116)$, and $\underline{\delta}^a_{MV} = (5.145, 5.128)$, where $\underline{\delta}^a_{MV}$ denotes the improved version of $\underline{\delta}_{MV}$ obtained by using Theorem 3.1. In this situation we recommend to use the estimator $\underline{\delta}^a_{MV}$.

4. A GENERALIZATION

In this section we extend some of the results obtained in Sections 2 and 3 to the $k\,(\geq 2)$ exponential populations and obtain the improved estimators for the UMVUE $\underline{\delta}_{MV}$.

Specifically, let $X_{i1}, X_{i2}, ..., X_{in_i}$ be a random sample of size n_i taken from the i^{th} exponential population $Ex(\mu, \sigma_i)$. The random variable X_{ij} has probability density function,

$$f(x_{ij}) = \frac{1}{\sigma_i} \exp\Big\{-\Big(\frac{x_{ij}-\mu}{\sigma_i}\Big)\Big\}, \qquad x_{ij} > \mu,\; -\infty < \mu < \infty,\; \sigma_i > 0\,, \qquad j = 1, 2, ..., n_i;\; i = 1, 2, ..., k\,.$$

We estimate the quantile vector $\underline{\theta} = (\theta_1, \theta_2, ..., \theta_k)$; where $\theta_i = \mu + \eta\sigma_i$ be the quantile of the i^{th} population, with respect to the loss function either

$$L_1(\underline{\alpha}, \underline{d}) = \sum_{i=1}^{k} (d_i - \theta_i)^2\,, \tag{4.1}$$

or

$$L_2(\underline{\alpha},\underline{d}) = \sum_{i=1}^{k}\left(\frac{d_i-\theta_i}{\sigma_i}\right)^2. \tag{4.2}$$

Let us denote $X_i = \min(X_{i1}, X_{i2}, ..., X_{in_i})$ and $Y_i = \frac{1}{n_i}\sum_{j=1}^{n_i} X_{ij}$. Further define $Z = \min(X_1, X_2, ..., X_k)$, and $T_i = Y_i - Z$; $i=1,2,...,k$. Here Y_i is the sample mean from the i^{th} population. Then $(Z, T_1, T_2, ..., T_k)$ is a complete sufficient statistic. Now the random variables Z and $\underline{T} = (T_1, T_2, ..., T_k)$ are independently distributed. The probability density function of Z is given by

$$f_Z(z) = a\exp\bigl(-a(z-\mu)\bigr)\,, \qquad z>\mu,\quad -\infty<\mu<\infty\,,$$

where $a = \sum_{i=1}^{k}\frac{n_i}{\sigma_i}$. The joint probability density function of $\underline{T} = (T_1, T_2, ..., T_k)$ is given by

$$f_{\underline{T}}(\underline{t}) = \frac{1}{a}\prod_{i=1}^{k}\left(\frac{n_i^{n_i}}{\sigma_i^{n_i}}\right)\prod_{i=1}^{k}\left(\frac{t_i^{n_i-1}}{\Gamma n_i}\right)\left[\sum_{i=1}^{k}\frac{n_i-1}{t_i}\right]\exp\left\{-\sum_{i=1}^{k} n_i t_i/\sigma_i\right\}, \qquad t_i>0\,.$$

It should be noted that the MLE $\underline{\delta}_{ML}$, modification to the MLE $\underline{\delta}_{MM}$ and the UMVUE $\underline{\delta}_{MV}$ can easily be obtained as

$$\underline{\delta}_{ML} = Z\underline{e} + \eta\underline{T}\,,$$

$$\underline{\delta}_{MM} = (Z-\hat{a}^{-1})\underline{e} + \eta\underline{T}$$

and

$$\underline{\delta}_{MV} = \bigl(Z + (\eta-1)\,T^*\bigr)\underline{e} + \eta\underline{T}\,,$$

where $\underline{e} = (1,1,...,1)_{1\times k}$ and $T^* = \bigl(\sum_{i=1}^{k}(n_i-1)\,T_i^{-1}\bigr)^{-1}$.

Consider the class of estimators for the quantile vector $\underline{\theta}$ as

$$D_{\mathbf{c}} = \Bigl\{\underline{\delta}_{\mathbf{c}}\colon\ \underline{\delta}_{\mathbf{c}} = (\delta_{c_1}, \delta_{c_2}, ..., \delta_{c_k});\ c_i\in\mathbb{R}\Bigr\}\,,$$

where we denote $\mathbf{c} = (c_1, c_2, ..., c_k)$, and

$$\delta_{c_j} = Z + \eta\, c_j T_j + (\eta-1)T^*; \qquad j=1,2,...,k\,.$$

It should be noted that, this class contains the estimator $\underline{\delta}_{MV}$ for $c_1 = c_2 = \cdots = c_k = 1$.

Now for the class of estimators $D_{\mathbf{c}}$ define

$$\mathbf{c}^* = \Bigl(\min\bigl\{\max(c_1,a_1),b_1\bigr\},\ ...,\ \min\bigl\{\max(c_k,a_k),b_k\bigr\}\Bigr) \tag{4.3}$$

and

$$\mathbf{c}_* = \Big(\min\big\{\max(c_1, c_1^+), d_1\big\}, \ldots, \min\big\{\max(c_k, c_k^+), d_k\big\}\Big), \tag{4.4}$$

where $a_j = \frac{\eta n_j(n_j-2)+1}{\eta n_j(n_j-1)}$, $b_j = \frac{n_j}{n_j+1}$, $d_j = \max\{a_j, b_j\}$, $c_j^+ = \hat{c}_j(\lambda_j^+)$, and $\lambda_j^+ = \big\{(n_j+1) - \sqrt{(n_j+1)^2 - 4\eta n_j}\big\}/2n_j$; $j = 1, 2, \ldots, k$. Next we have the following inadmissibility result for estimators in the class $D_{\mathbf{c}}$.

Theorem 4.1. *Let $D_{\mathbf{c}}$ be the class of estimators for the quantile vector $\underline{\theta}$, and define the functions $\mathbf{c}^*$ and $\mathbf{c}_*$ as in (4.3) and (4.4) respectively. Let the loss function be* (4.1).

- **(i)** *The estimator $\underline{\delta}_{\mathbf{c}}$ is inadmissible and is improved by $\underline{\delta}_{\mathbf{c}^*}$ if $\mathbf{c} \neq \mathbf{c}^*$, when $\eta \geq 1$.*
- **(ii)** *The estimator $\underline{\delta}_{\mathbf{c}}$ is inadmissible and is improved by $\underline{\delta}_{\mathbf{c}_*}$ if $\mathbf{c} \neq \mathbf{c}_*$ when $0 < \eta < 1$.*

Proof: The proof is similar to the proof of the Theorem 2.1. □

The class of estimators $D_{\mathbf{c}}$ contains the UMVUE of the quantile vector $\underline{\theta}$ when $c_i = 1$; $i = 1, 2, \ldots, k$. Consequently, the UMVUE $\underline{\delta}_{MV}$ is inadmissible. Let $p_1 = \min\{\frac{1}{n_1}, \ldots, \frac{1}{n_k}\}$, $p_2 = \max\{\frac{1}{n_1}, \ldots, \frac{1}{n_k}\}$, $q_1 = \min\{\frac{n_1+1}{2n_1}, \ldots, \frac{n_k+1}{2n_k}\}$ and $q_2 = \max\{\frac{n_1+1}{2n_1}, \ldots, \frac{n_k+1}{2n_k}\}$.

Theorem 4.2. *Let the loss function be (4.1).*

- **(i)** *If $\eta \geq 1$, then the uniformly minimum variance unbiased estimator $\underline{\delta}_{MV}$ for the quantile vector $\underline{\theta}$ is inadmissible and is improved by the estimator $\underline{\delta}_{IMV} = (\delta_{b_1}, \ldots, \delta_{b_k})$. Further the class $\{\underline{\delta}_{\mathbf{c}} = (\delta_{c_1}, \ldots, \delta_{c_k})\colon a_j \leq c_j \leq b_j;\ j = 1, 2, \ldots, k\}$ is essentially complete in $D_{\mathbf{c}}$.*
- **(ii)** *If $q_2 \leq \eta < 1$, then the estimator $\underline{\delta}_{MV}$ is inadmissible and is improved by $\underline{\delta}_{IMV} = (\delta_{b_1}, \ldots, \delta_{b_k})$. The class of estimators $\{\underline{\delta}_{\mathbf{c}} = (\delta_{c_1}, \ldots, \delta_{c_k})\colon c_j^+ \leq c_j \leq b_j;\ j = 1, 2, \ldots, k\}$ is essentially complete in $D_{\mathbf{c}}$.*
- **(iii)** *If $p_2 \leq \eta \leq q_1$, then the estimator $\underline{\delta}_{MV}$ is inadmissible and is improved by $\underline{\delta}_{IMV} = (\delta_{a_1}, \ldots, \delta_{a_k})$. The class of estimators $\{\underline{\delta}_{\mathbf{c}} = (\delta_{c_1}, \ldots, \delta_{c_k})\colon c_j^+ \leq c_j \leq a_j;\ j = 1, 2, \ldots, k\}$ is essentially complete in $D_{\mathbf{c}}$.*
- **(iv)** *If $0 \leq \eta < p_1$, then the estimator $\underline{\delta}_{MV}$ is admissible in the class D. The class of estimators $\{\underline{\delta}_{\mathbf{c}} = (\delta_{c_1}, \ldots, \delta_{c_k})\colon c_j^+ \leq c_j \leq a_j;\ j = 1, 2, \ldots, k\}$ is essentially complete in $D_{\mathbf{c}}$.*
- **(v)** *Let $p_1 < \eta < p_2$ and $(l_1, l_2, \ldots, l_k)$ be a permutation of $(1, 2, \ldots, k)$ such that $1/n_{l_1} < \eta$, ..., $1/n_{l_p} < \eta$, and $1/n_{l_{p+1}} \geq \eta$, ..., $1/n_{l_k} \geq \eta$.*

Then the estimator $\underline{\delta}_{MV}$ *is inadmissible and is improved by* $\underline{\delta}_{IMV} = (\delta_{a_{l_1}}, ..., \delta_{a_{l_p}}, \delta_{c_{l_{p+1}}}, ..., \delta_{c_{l_k}})$, *when* $\eta < \frac{n_{l_1}+1}{2n_{l_1}}, ..., \eta < \frac{n_{l_p}+1}{2n_{l_p}}$, *and improved by* $\underline{\delta}_{IMV} = (\delta_{b_{l_1}}, ..., \delta_{b_{l_p}}, \delta_{c_{l_{p+1}}}, ..., \delta_{c_{l_k}})$, *when* $\eta \geq \frac{n_{l_1}+1}{2n_{l_1}}, ...,$ $\eta \geq \frac{n_{l_p}+1}{2n_{p+1}}$ *where* $c_{l_{p+1}} = c_{l_{p+2}} = \cdots = c_{l_k} = 1$.

(vi) *Let* $p_1 < \eta < p_2$ *and* $(l_1, l_2, ..., l_k)$ *be a permutation of* $(1, 2, ..., k)$ *such that* $1/n_{l_1} \geq \eta$, ..., $1/n_{l_p} \geq \eta$, *and* $1/n_{l_{p+1}} < \eta$, ..., $1/n_{l_k} < \eta$. *Then the estimator* $\underline{\delta}_{MV}$ *is inadmissible and is improved by* $\underline{\delta}_{IMV} = (\delta_{c_{l_1}}, ..., \delta_{c_{l_p}}, \delta_{a_{l_{p+1}}}, ..., \delta_{a_{l_k}})$, *when* $\eta < \frac{n_{l_{p+1}}+1}{2n_{l_{p+1}}}$, ..., $\eta < \frac{n_{l_k}+1}{2n_{l_k}}$, *and improved by* $\underline{\delta}_{IMV} = (\delta_{c_{l_1}}, ..., \delta_{c_{l_p}}, \delta_{b_{l_{p+1}}}, ..., \delta_{b_{l_k}})$, *when* $\eta \geq \frac{n_{l_{p+1}}+1}{2n_{l_{p+1}}}$, ..., $\eta \geq \frac{n_{l_k}+1}{2n_k}$, *where* $c_{l_1} = c_{l_2} = \cdots = c_{l_p} = 1$.

(vii) *Let* $q_1 < \eta < q_2$ *and* $(l_1, l_2, ..., l_k)$ *be a permutation of* $(1, 2, ..., k)$ *such that* $\frac{n_{l_1}+1}{2n_{l_1}} < \eta$, $\frac{n_{l_2}+1}{2n_{l_2}} < \eta$, ..., $\frac{n_{l_p}+1}{2n_{l_p}} < \eta$ *and* $\frac{n_{l_{p+1}}+1}{2n_{l_{p+1}}} \geq \eta$, $\frac{n_{l_{p+2}}+1}{2n_{l_{p+2}}} \geq \eta$, ..., $\frac{n_{l_k}+1}{2n_{l_k}} \geq \eta$. *The estimator* $\underline{\delta}_{MV}$ *is inadmissible and is improved by* $\underline{\delta}_{IMV} = (\delta_{b_{l_1}}, ..., \delta_{b_{l_p}}, \delta_{a_{l_{p+1}}}, ..., \delta_{a_{l_k}})$, *when* $\eta > 1/n_{l_{p+1}}$, $\eta > 1/n_{l_{p+2}}$, ..., $\eta > 1/n_{l_k}$ *and by* $\underline{\delta}_{IMV} = (\delta_{b_{l_1}}, ..., \delta_{b_{l_p}}, \delta_{c_{l_{p+1}}}, ..., \delta_{c_{l_k}})$, *when* $\eta \leq 1/n_{l_{p+1}}$, $\eta \leq 1/n_{l_{p+2}}$, ..., $\eta \leq 1/n_{l_k}$, *where* $c_{l_{p+1}} = c_{l_{p+2}} = \cdots = c_{l_k} = 1$.

(viii) *Let* $q_1 < \eta < q_2$ *and* $(l_1, l_2, ..., l_k)$ *be a permutation of* $(1, 2, ..., k)$ *such that* $\frac{n_{l_1}+1}{2n_{l_1}} \geq \eta$, $\frac{n_{l_2}+1}{2n_{l_2}} \geq \eta$, ..., $\frac{n_{l_p}+1}{2n_{l_p}} \geq \eta$ *and* $\frac{n_{l_{p+1}}+1}{2n_{l_{p+1}}} < \eta$, $\frac{n_{l_{p+2}}+1}{2n_{l_{p+2}}} < \eta$, ..., $\frac{n_{l_k}+1}{2n_{l_k}} < \eta$. *The estimator* $\underline{\delta}_{MV}$ *is inadmissible and is improved by* $\underline{\delta}_{IMV} = (\delta_{a_{l_1}}, ..., \delta_{a_{l_p}}, \delta_{b_{l_{p+1}}}, ..., \delta_{b_{l_k}})$, *when* $\eta > 1/n_{l_1}$, $\eta > 1/n_{l_2}$, ..., $\eta > 1/n_{l_p}$ *and by* $\underline{\delta}_{IMV} = (\delta_{c_{l_1}}, ..., \delta_{c_{l_p}}, \delta_{b_{l_{p+1}}}, ..., \delta_{b_{l_k}})$, *when* $\eta \leq 1/n_{l_1}$, $\eta \leq 1/n_{l_2}$, ..., $\eta \leq 1/n_{l_p}$, *where* $c_{l_1} = c_{l_2} = \cdots = c_{l_p} = 1$.

Applying the above Theorem 4.2 we can obtain the estimator which improves upon $\underline{\delta}_{MV}$. However, we have obtained the expressions for some specific values of η. One can easily write the estimator for other choices of η:

$$\underline{\delta}_{IMV} = \begin{cases} \big(Z + (\eta-1)T^*\big)\underline{e} + \eta\underline{B}\,, & \text{if } \eta \geq q_2\,, \\ \big(Z + (\eta-1)T^*\big)\underline{e} + \eta\underline{A}\,, & \text{if } p_2 < \eta \leq q_1\,, \end{cases}$$

where $\underline{A} = (A_1, A_2, ..., A_k)$; $A_i = a_i T_i$ and $\underline{B} = (B_1, B_2, ..., B_k)$; $B_i = b_i T_i$; $i = 1, 2, ..., k$.

Next we generalize the results obtained in Theorem 3.1 and Theorem 3.2. Let us consider the affine group of transformations, $G_A = \{g_{a,b} \colon g_{a,b}(x) = ax + b$, $a > 0$, $b \in \mathbb{R}$. Under the transformation $g_{a,b}$, we have $Z \to aZ + b$, $T_i \to aT_i$, $\sigma_i \to a\sigma_i$, $\mu \to a\mu + b$, and $\theta_i = \mu + \eta\sigma_i \to a\theta_i + b$; $i = 1, 2, ..., k$. So $\underline{\theta} \to a\underline{\theta} + b\underline{e}$, where $\underline{e} = (1, 1, ..., 1)_{1\times k}$. Under this transformation the problem remains invariant if we choose the loss function (4.2), and the form of an affine equivariant estimator is obtained as

$$\begin{aligned} \underline{\delta}(Z, T_1, T_2, ..., T_k) &= Z\underline{e} + T_1 \underline{\Psi}(\underline{W}) \\ &= \underline{\delta}_{\Psi}\,, \quad \text{say}\,, \end{aligned} \tag{4.5}$$

where $\underline{W} = (W_2, W_3, ..., W_k)$ and $W_i = \frac{T_i}{T_1}$; $i = 2, 3, ..., k$.

Consider the conditional risk function:

$$R(\underline{\delta}_{\Psi}, \underline{\alpha} \,|\, \underline{W}) = \sum_{i=1}^{k} E\left\{ \left(\frac{Z + T_1 \Psi_i(\underline{W}) - \theta_i}{\sigma_i} \right)^2 \,\Big|\, \underline{W} \right\}. \tag{4.6}$$

It is easy to observe that the above conditional risk is a convex function in each Ψ_i and hence the sum. The minimizing choices for each Ψ_i is obtained as

$$\hat{\Psi}_i = -\frac{E(Z - \theta_i)\, E(T_1 \,|\, \underline{W})}{E(T_1^2 \,|\, \underline{W})}\,; \qquad i = 1, 2, ..., k\,. \tag{4.7}$$

After evaluating the conditional expectations and simplifying we have the minimizing choice of Ψ_i as

$$\hat{\Psi}_i = \frac{1}{\sum_{j=1}^{k} n_j} \left[\eta\sigma_i - a^{-1}\right] \left[\frac{n_1}{\sigma_1} + \sum_{j=2}^{k} \frac{n_j w_j}{\sigma_j}\right]; \qquad i = 1, 2, ..., k\,. \tag{4.8}$$

To apply the Brewster and Zidek technique we need to find the supremum and infimum of each $\hat{\Psi}_i$ with respect to $\underline{\sigma} = (\sigma_1, ..., \sigma_k)$ for fixed values of η, n_i and $\underline{W}$. We are not able to obtain the supremum and infimum for each $\hat{\Psi}_i$ for the case $k\,(\geq 3)$. However, for the first component $\hat{\Psi}_1$, Sharma and Kumar [19] obtained the bounds for equal sample sizes. We feel that the lower bounds for other components will be finite. Since we are not able to derive the bounds for the case $k\,(\geq 3)$, it could not be possible to provide the inadmissibility result for $k\,(\geq 3)$ populations. It will be interesting to obtain the bounds for the case $k\,(\geq 3)$ and obtain improved estimators better than $\underline{\delta}_{MV}$.

5. NUMERICAL COMPARISONS

In this section, we carry out a detailed simulation study to numerically compare the risk functions of various estimators proposed in previous sections for the quantile vector $\underline{\theta}$ for the case $k = 2$. Specifically, we have proposed some baseline estimators such as $\underline{\delta}_{ML}$, $\underline{\delta}_{MM}$ and $\underline{\delta}_{MV}$ for $\underline{\theta}$. An improved estimator $\underline{\delta}_{IMV}$ which dominates $\underline{\delta}_{MV}$ has been obtained in Section 2 for the case $\eta \geq p_1$. From the Remark 3.1, it is quite evident that, we only consider the estimator $\underline{\delta}_{MV}$ and obtain its improved version by using the Theorem 3.1, which we denote as

$\underline{\delta}^a_{MV}$ for the case $0 < \eta < p_2$. For numerically comparing the risk functions of all these estimators for $\underline{\theta}$, we use Monte-Carlo simulation procedure. We have generated 10,000 random samples each from two exponential populations $Exp(\mu, \sigma_1)$ and $E(\mu, \sigma_2)$ respectively. Here μ is the common location parameter and σ_1, σ_2 are different scale parameters. The loss function is taken as the sum of the quadratic losses (1.2). It should be noted that, with respect to the loss (1.2), the risk functions of each estimator is a function of only $\tau = \sigma_1/\sigma_2 > 0$ for fixed values of η and sample sizes. A massive simulation study has been carried out to see the behavior of the risk functions and the performance of each estimator for the quantile vector $\underline{\theta}$. The error of the simulation has been checked and it is quite satisfactory (up to order of 10^{-3}). We have also calculated the percentage of relative risk performances for each estimator with respect to the baseline estimator $\underline{\delta}_{ML}$. For this purpose we define the equation,

$$R_{MM} = \left(\frac{\underline{\delta}_{ML} - \underline{\delta}_{MM}}{\underline{\delta}_{ML}}\right) \times 100 \ , \qquad R_{MV} = \left(\frac{\underline{\delta}_{ML} - \underline{\delta}_{MV}}{\underline{\delta}_{ML}}\right) \times 100 \ ,$$

$$R_{MVA} = \left(\frac{\underline{\delta}_{ML} - \underline{\delta}^a_{MV}}{\underline{\delta}_{ML}}\right) \times 100 \ , \qquad R_{IMV} = \left(\frac{\underline{\delta}_{ML} - \underline{\delta}_{IMV}}{\underline{\delta}_{ML}}\right) \times 100 \ .$$

For illustration purpose, we choose some specific values of η and n_1, n_2. Though the values of τ can be from 0 to ∞, we choose the values up to 5 to avoid simulation error. The percentage of relative risk improvements of all the estimators over the MLE has been tabulated in Tables 1 to 3. In Table 1, we have tabulated the percentage of relative risk values for equal sample sizes whereas Tables 2, 3 gives for unequal sample sizes. In each table, the first row gives the various choices of η. We have taken conveniently the values of η as 0.05 and 2.50. The first column represents the values of τ which ranges from 0 to 5. Further, for each value of η, there corresponds three columns (columns 1, 2, 3 correspond to $\eta = 0.05$ and columns 4, 5, 6 correspond to $\eta = 2.50$). For each value of τ, there corresponds three values of percentage of relative risk values. These three values corresponds to three different pairs of sample sizes, for example in Table 1, the percentage of relative risk values have been tabulated for the sample sizes $(5, 5)$, $(10, 10)$ and $(15, 15)$. Similarly in Tables 2 and 3, the percentage of relative risk performances have been tabulated for the sample sizes $(3, 7)$, $(5, 10)$, $(10, 15)$ and $(7, 3)$, $(10, 5)$, $(15, 10)$ respectively.

The following conclusions can be drawn from our simulation study as well as from the Tables 1, 2 and 3.

- **(i)** It is observed that as the sample sizes (n_1 and n_2) increase the risk values decrease for fixed value of η.
- **(ii)** For $0 < \eta \le p_1$, the estimator $\underline{\delta}^a_{MV}$ has the least risk for almost all values of the parameters except few values where the estimator $\underline{\delta}_{MM}$ performs marginally better. The percentage of relative risk improvement has been noticed and is near 50%.

(iii) For $\eta > p_2$, the estimator $\underline{\delta}_{IMV}$ performs the best and the percentage of relative risk improvement is near 21%. However, the performance decreases as the sample sizes increase.

(iv) When η lies in the interval $[p_1, p_2]$, the estimators $\underline{\delta}_{IMV}$ and $\underline{\delta}^a_{MV}$ compete well with each other. In fact for small values of τ, the estimator $\underline{\delta}_{IMV}$ performs better compared to $\underline{\delta}^a_{MV}$ whereas for larger values of τ, the estimator $\underline{\delta}^a_{MV}$ performs better. However, the estimator $\underline{\delta}_{MM}$ has the best percentage of relative risk improvement for this choice of η.

(v) For $\eta = 1$, (the problem reduces to simultaneous estimation of means of two exponential populations) the estimators $\underline{\delta}_{ML}$ and $\underline{\delta}_{MV}$ are equal and it is also noticed that the performance of $\underline{\delta}_{IMV}$ is the best.

(vi) The numerical study also shows that the estimator $\underline{\delta}^a_{MV}$ improves upon $\underline{\delta}_{MV}$, which agrees with the Theorem 3.1. Further the estimator $\underline{\delta}_{MV}$ is improved by $\underline{\delta}_{IMV}$ which also agrees with the Theorem 2.1.

(vii) Similar type of observations were made for other combinations of η and sample sizes during our simulation study.

(viii) On the basis of our simulation study and theoretical findings, we recommend using the estimator $\underline{\delta}^a_{MV}$ when $\eta < p_1$ and $\underline{\delta}_{IMV}$ when $\eta \geq p_2$, whereas we recommend to use $\underline{\delta}_{MM}$ for η lying in the interval $[p_1, p_2]$.

6. CONCLUDING REMARKS

In this paper we have considered the estimation of the quantile vector $\underline{\theta} = (\theta_1, \theta_2, ..., \theta_k)$ of $k\,(\geq 2)$ exponential populations with respect to the sum of the quadratic loss functions or the sum of the squared error losses. We first proposed estimators for $\underline{\theta}$ which are based on some baseline estimators for each component θ_i, such as MLE and UMVUE. We have constructed a class containing the estimator based on UMVUE of θ_i. Some techniques for improving estimators have been used to obtain estimators which dominate the UMVUE of $\underline{\theta}$. Further an admissible class has been obtained within the class. Next we have introduced the concept of invariance to our model and derive sufficient conditions for improving estimators which are equivariant under the location and affine group of transformations for the case $k = 2$. The inadmissibility result for the case $k\,(\geq 3)$ populations is not available. Finally, we have conducted a simulation study to numerically compare the risk functions of all the proposed estimators and recommended their use in practice. It may be noted that the simultaneous estimation of quantiles of $k\,(\geq 2)$ exponential populations has not been studied in the literature before.

Table 1: Relative risk performances of various estimators of exponential quantiles for $(n_1, n_2) = (5, 5), (10, 10), (15, 15)$.

$\eta \rightarrow$	0.05			2.5		
$\tau \downarrow$	R_{MM}	R_{MV}	R_{MVA}	R_{MM}	R_{MV}	R_{IMV}
0.25	43.037 46.808 48.318	42.598 46.703 48.335	43.374 46.834 48.337	0.956 0.733 0.558	− 6.395 − 3.383 − 2.280	12.066 07.430 04.900
0.50	45.155 46.311 46.890	45.148 46.313 46.924	45.766 46.385 46.924	0.904 0.422 0.284	− 5.236 − 2.411 − 1.579	14.091 07.394 05.167
0.75	45.279 47.002 46.913	45.535 47.151 46.967	46.122 47.170 46.967	0.723 0.427 0.349	− 4.625 − 2.288 − 1.602	14.678 07.784 05.419
1.00	45.610 46.767 46.952	45.619 46.899 46.988	46.350 46.914 46.988	0.888 0.283 0.196	− 4.906 − 1.975 − 1.335	15.309 07.316 04.996
1.25	45.067 46.928 46.842	44.954 47.097 46.877	45.740 47.114 46.877	0.887 0.388 0.357	− 4.856 − 2.220 − 1.583	14.926 07.611 05.452
1.50	44.834 46.555 47.772	45.058 46.651 47.818	45.588 46.695 47.818	0.751 0.411 0.231	− 4.758 − 2.283 − 1.442	14.388 07.798 05.076
1.75	44.454 46.699 46.982	44.741 46.759 47.000	45.233 46.802 47.000	0.771 0.398 0.197	− 4.958 − 2.331 − 1.428	14.146 07.490 04.570
2.00	44.042 47.795 46.428	44.125 47.894 46.478	44.621 47.959 46.478	0.484 0.446 0.388	− 4.520 − 2.465 − 1.764	13.347 07.531 05.469
2.25	43.780 47.813 46.457	43.699 47.923 46.483	44.397 47.982 46.483	0.604 0.424 0.384	− 4.902 − 2.473 − 1.785	13.734 07.530 05.191
2.50	45.520 46.369 48.159	45.264 46.472 48.209	45.949 46.514 48.210	0.884 0.509 0.323	− 5.517 − 2.688 − 1.729	14.398 06.946 05.094
2.75	44.021 45.821 46.457	43.564 45.882 46.476	44.495 45.934 46.477	0.677 0.623 0.349	− 5.304 − 2.921 − 1.812	13.024 07.492 04.967
3.00	44.332 47.501 47.773	44.338 47.549 47.815	44.891 47.618 47.818	1.007 0.562 0.363	− 5.999 − 2.880 − 1.879	13.630 07.435 05.051
3.25	44.406 46.898 47.658	43.960 46.943 47.686	44.874 47.001 47.689	0.882 0.361 0.249	− 5.972 − 2.632 − 1.684	12.737 06.610 04.841
3.50	43.503 47.113 47.248	43.082 47.175 47.282	43.878 47.218 47.284	1.084 0.356 0.295	− 6.421 − 2.651 − 1.828	13.541 06.900 04.781
3.75	43.657 46.720 47.962	43.236 46.767 47.972	43.999 46.771 47.975	0.856 0.382 0.363	− 6.091 − 2.740 − 1.968	12.500 07.112 04.365
4.00	43.683 47.308 47.804	43.505 47.372 47.831	44.188 47.399 47.836	1.107 0.412 0.229	− 6.681 − 2.830 − 1.748	12.581 06.585 04.923
4.50	43.373 47.327 47.789	43.061 47.423 47.838	43.748 47.448 47.843	1.053 0.451 0.520	− 6.755 − 2.971 − 2.294	12.813 06.532 04.967
5.00	44.398 45.883 47.180	43.750 45.957 47.197	44.536 45.990 47.212	1.066 0.459 0.519	− 6.961 − 3.029 − 2.339	11.409 06.346 04.823

Table 2: Relative risk performances of various estimators of exponential quantiles for $(n_1, n_2) = (3,7), (5,10), (10,15)$.

$\eta \rightarrow$	0.05			2.5		
$\tau \downarrow$	R_{MM}	R_{MV}	R_{MVA}	R_{MM}	R_{MV}	R_{IMV}
0.25	41.906 45.474 47.309	41.323 45.569 47.391	43.081 45.924 47.405	1.669 1.036 0.272	− 9.291 − 5.772 − 2.487	16.740 10.976 05.275
0.50	42.950 46.733 47.516	43.248 46.888 47.558	44.022 47.201 47.599	1.333 0.709 0.524	− 6.102 − 3.726 − 2.323	19.997 12.461 06.857
0.75	44.397 46.547 47.023	44.332 46.741 47.052	45.282 46.994 47.062	0.821 0.511 0.308	− 4.296 − 2.926 − 1.779	20.276 12.607 06.602
1.00	45.131 46.561 47.745	45.321 46.768 47.789	46.071 46.934 47.790	0.644 0.571 0.180	− 3.677 − 2.775 − 1.442	20.487 13.186 06.457
1.25	45.226 46.731 47.602	45.386 46.724 47.710	46.131 46.896 47.710	0.721 0.500 0.357	− 3.704 − 2.604 − 1.713	20.581 12.887 07.150
1.50	44.199 45.620 47.137	44.099 45.780 47.204	44.772 45.906 47.204	0.544 0.459 0.189	− 3.329 − 2.490 − 1.450	20.346 12.689 06.663
1.75	44.278 44.896 47.072	44.194 44.962 47.097	44.857 45.059 47.097	0.453 0.437 0.192	− 3.072 − 2.428 − 1.462	20.837 12.735 06.699
2.00	43.354 46.011 47.194	43.583 45.911 47.230	44.053 46.086 47.232	0.572 0.374 0.261	− 3.335 − 2.328 − 1.589	20.450 12.562 06.637
2.25	44.806 46.415 46.523	44.845 46.399 46.534	45.397 46.520 46.535	0.522 0.438 0.248	− 3.291 − 2.407 − 1.596	19.334 13.214 06.240
2.50	44.570 45.588 46.890	44.353 45.585 46.951	44.797 45.687 46.953	0.440 0.433 0.278	− 3.195 − 2.389 − 1.630	19.147 13.896 06.169
2.75	45.507 45.795 46.556	45.258 45.688 46.616	45.759 45.817 46.618	0.529 0.459 0.293	− 3.319 − 2.505 − 1.688	20.592 12.584 06.672
3.00	43.824 45.929 46.921	43.592 45.869 46.927	44.095 45.975 46.936	0.466 0.367 0.293	− 3.226 − 2.334 − 1.705	19.385 12.770 06.848
3.25	44.325 46.165 47.061	44.097 46.129 47.054	44.547 46.258 47.064	0.399 0.318 0.289	− 3.162 − 2.313 − 1.752	19.697 11.657 06.671
3.50	44.469 47.253 46.078	44.156 47.241 46.081	44.536 47.366 46.095	0.503 0.242 0.236	− 3.380 − 2.167 − 1.670	19.444 12.236 06.318
3.75	45.216 45.286 46.998	44.809 45.188 47.056	45.341 45.343 47.069	0.401 0.340 0.358	− 3.146 − 2.374 − 1.832	19.755 11.737 06.970
4.00	44.480 46.173 46.851	43.966 46.131 46.880	44.530 46.227 46.895	0.443 0.401 0.344	− 3.305 − 2.472 − 1.844	19.008 12.908 07.032
4.50	45.394 45.033 46.975	44.937 44.991 46.970	45.334 45.061 46.978	0.217 0.352 0.284	− 2.858 − 2.414 − 1.774	18.840 12.076 06.626
5.00	43.720 45.965 46.456	43.404 45.931 46.492	43.822 46.002 46.496	0.507 0.404 0.314	− 3.385 − 2.543 − 1.870	19.881 12.480 06.477

Table 3: Relative risk performances of various estimators of exponential quantiles for $(n_1, n_2) = (7,3), (10,5), (15,10)$.

$\eta \rightarrow$	0.05			2.5		
$\tau \downarrow$	R_{MM}	R_{MV}	R_{MVA}	R_{MM}	R_{MV}	R_{IMV}
0.25	45.446 45.940 47.235	44.948 45.996 47.222	45.439 46.036 47.237	0.435 0.494 0.266	− 3.278 − 2.602 − 1.737	19.087 13.201 06.439
0.50	43.214 46.619 47.152	43.298 46.648 47.149	43.805 46.770 47.151	0.455 0.378 0.469	− 3.153 − 2.303 − 1.929	19.717 12.980 07.236
0.75	44.330 45.780 47.580	44.142 45.801 47.639	44.969 45.959 47.639	0.594 0.427 0.319	− 3.472 − 2.390 − 1.657	19.887 13.113 06.724
1.00	44.521 46.106 46.928	44.644 46.110 46.982	45.443 46.330 46.984	0.629 0.347 0.265	− 3.662 − 2.378 − 1.605	20.348 12.591 06.852
1.25	45.323 46.781 47.166	45.618 47.000 47.143	46.424 47.221 47.149	0.850 0.578 0.292	− 4.282 − 2.945 − 1.690	20.591 12.776 06.746
1.50	43.385 46.763 47.490	43.322 46.766 47.5304	44.350 47.085 47.538	1.014 0.772 0.456	− 4.806 − 3.466 − 2.055	21.011 13.435 07.111
1.75	44.075 47.294 47.159	44.128 47.384 47.170	45.052 47.732 47.196	1.306 0.515 0.443	− 5.750 − 3.208 − 2.137	20.317 12.078 06.862
2.00	42.991 46.865 47.128	43.446 47.210 47.208	44.243 47.448 47.224	0.968 0.730 0.409	− 5.305 − 3.761 − 2.164	19.192 12.445 06.537
2.25	43.771 46.120 46.895	44.121 46.198 46.937	44.938 46.525 46.979	1.630 0.912 0.417	− 7.037 − 4.282 − 2.228	20.335 12.849 06.382
2.50	43.007 46.586 47.448	42.970 46.613 47.503	44.183 47.026 47.556	1.373 0.698 0.358	− 6.872 − 4.150 − 2.244	19.011 11.695 06.047
2.75	43.741 46.123 48.110	43.624 46.068 48.221	44.675 46.504 48.257	1.554 0.913 0.480	− 7.584 − 4.627 − 2.523	18.337 12.069 06.138
3.00	43.166 45.193 47.727	42.889 45.383 47.848	44.267 45.637 47.875	1.744 0.918 0.511	− 8.175 − 4.830 − 2.656	18.991 11.355 06.237
3.25	42.960 45.672 47.430	43.097 45.976 47.459	44.335 46.113 47.546	1.845 0.888 0.527	− 8.768 − 5.019 − 2.755	18.520 11.025 06.115
3.50	41.211 45.775 46.820	40.899 45.681 46.848	42.355 46.111 46.931	1.603 1.078 0.591	− 8.551 − 5.463 − 2.955	17.837 11.228 05.738
3.75	42.765 45.668 48.711	42.426 45.778 48.813	44.002 46.103 48.853	1.943 1.048 0.475	− 9.507 − 5.678 − 2.794	18.044 10.862 05.613
4.00	41.437 45.805 47.696	40.853 45.877 47.710	42.613 46.266 47.768	2.086 1.226 0.578	− 10.158 − 6.067 − 3.029	17.591 10.923 05.695
4.50	42.217 45.568 48.148	41.511 45.667 48.199	43.183 46.074 48.243	2.007 1.177 0.640	− 10.418 − 6.297 − 3.251	17.181 10.417 05.778
5.00	40.642 45.277 47.350	39.702 45.013 47.432	41.661 45.486 47.445	2.120 1.145 0.665	− 11.496 − 6.369 − 3.375	16.171 10.399 05.678

APPENDIX

Proof of Theorem 2.1: In order to prove the theorem we use the orbit-by-orbit improvement technique of Brewster and Zidek [2].

Consider the risk function of $\underline{\delta}_{\mathbf{c}}$ with respect to the loss function (1.1),

$$R(\underline{\alpha},\underline{\delta}_{\mathbf{c}}) = E\Big[Z+\eta c_1 T_1 + (\eta-1)T^* - \mu - \eta\sigma_1\Big]^2 + E\Big[Z+\eta c_2 T_2 + (\eta-1)T^* - \mu - \eta\sigma_2\Big]^2 . \tag{A.1}$$

It can be easily seen that the above risk (A.1) is a convex function in both c_1 and c_2. After some calculations, the minimizing choices for c_1 and c_2 are obtained as

$$\hat{c}_j(\underline{\alpha}) = \frac{(\mu+\eta\sigma_j)ET_j - E(ZT_j) - (\eta-1)E(T_j T^*)}{\eta E T_j^2}\,; \qquad j=1,2\,. \tag{A.2}$$

Let $\lambda_j = (\sigma_j a)^{-1}$, and using this we obtain the minimizing choice of each c_j as

$$\hat{c}_j(\lambda_j) = \frac{n_j\big(\eta - 2\eta\lambda_j + \lambda_j^2\big)}{\eta\big(1+n_j-2n_j\lambda_j\big)}\,; \qquad j=1,2\,. \tag{A.3}$$

To apply the orbit-by-orbit improvement technique of Brewster and Zidek [2], we need to get the supremum and infimum values of $\hat{c}_1$ and $\hat{c}_2$ with respect to λ_j and for fixed η. It is easy to see that $0<\lambda_j<\frac{1}{n_j}$. We consider the following three separate cases.

Case-(I): Let $\eta \geq \max\big((n_1{+}1)^2/4n_1,\,(n_2{+}1)^2/4n_2\big)$. Differentiating $\hat{c}_j(\lambda_j)$ with respect to λ_j we have $\frac{d\hat{c}_j}{d\lambda_j} = \frac{-2n_j\left(n_j\lambda_j^2-\lambda_j(n_j+1)+\eta\right)}{\eta(n_j+1-2n_j\lambda_j)^2}$; $j=1,2$. It is easy to observe that the derivative is $g(\lambda_j) = -n_j\lambda_j^2 + \lambda_j(n_j+1) - \eta$ multiplied by a positive factor. Now $g(\lambda_j)$ is a concave function of $\lambda_j; j=1,2$. The maximum value is attained at $\lambda_j = (n_j+1)/2n_j < 1/n_j$. The maximum value is $(n_j+1)^2/4n_j - \eta < 0$. This implies $g(\lambda_j)<0$ for $0<\lambda_j<\frac{1}{n_j}$; $j=1,2$. Hence the function $\hat{c}_j(\lambda_j)$ is decreasing with respect to λ_j. Hence we have

$$\inf_{0<\lambda_j\leq\frac{1}{n_j}} \hat{c}_j(\lambda_j) = \hat{c}_j(1/n_j) = \frac{n_j\eta(n_j-2)+1}{n_j\eta(n_j-1)} = a_j \quad \text{(say)}$$

and

$$\sup_{0<\lambda_j\leq\frac{1}{n_j}} \hat{c}_j(\lambda_j) = \hat{c}_j(0) = \frac{n_j}{n_j+1} = b_j\,; \quad j=1,2\,, \quad \text{(say)}\,.$$

Case-(II): Let $1 \leq \eta < \min\big\{(n_1{+}1)^2/4n_1,\ (n_2{+}1)^2/4n_2\big\}$. It is easy to see that the maximum value of $g(\lambda_j)$ is positive. The equation $g(\lambda_j) = 0$, has two real roots say $\lambda_j^- = \frac{(n_j+1)+\sqrt{(n_j+1)^2-4\eta n_j}}{2n_j}$ and $\lambda_j^+ = \frac{(n_j+1)-\sqrt{(n_j+1)^2-4\eta n_j}}{2n_j}$; $j=1,2$. It is also noticed that, these two roots are outside the interval $\big(0, \frac{1}{n_j}\big]$. Hence for $0 < \lambda_j \leq \frac{1}{n_j}$ the function $g(\lambda_j) < 0$. This implies that the function $\hat{c}_j(\lambda_j)$ is decreasing in the concerned interval. Hence we have

$$\inf_{0<\lambda_j\leq\frac{1}{n_j}} \hat{c}_j(\lambda_j) = \hat{c}_j(1/n_j) = \frac{n_j\eta(n_j-2)+1}{n_j\eta(n_j-1)} = a_j$$

and

$$\sup_{0<\lambda_j\leq\frac{1}{n_j}} \hat{c}_j(\lambda_j) = \hat{c}_j(0) = \frac{n_j}{n_j+1} = b_j\,; \quad j=1,2\,,$$

Case-(III): Let $0 \leq \eta < 1$. For this case it can be observed that the root λ_j^- is outside the concerned interval, but λ_j^+ is inside the interval $\big(0, \frac{1}{n_j}\big]$. Also $\hat{c}_j''(\lambda_j^+) > 0$ and $\hat{c}_j''(\lambda_j^-) < 0$, hence λ_j^- is a point of local maxima and λ_j^+ is a point of local minima. Hence the function $g(\lambda_j) < 0$ in the interval $(0, \lambda_j^+]$ and $g(\lambda_j) \geq 0$ in the interval $(\lambda_j^+, 1/n_j]$. Thus the function $\hat{c}_j(\lambda_j)$ is decreasing in the interval $(0, \lambda_j^+]$ and increasing in the interval $(\lambda_j^+, 1/n_j]$. We have

$$\inf_{0<\lambda_j<\frac{1}{n_j}} \hat{c}_j(\lambda_j) = \hat{c}_j(\lambda_j^+) = {c_j}^+\,,$$

and

$$\sup_{0<\lambda_j<\frac{1}{n_j}} \hat{c}_j(\lambda_j) = \max\Big\{\hat{c}_j(0), \hat{c}_j(1/n_j)\Big\} = \max\Big\{\frac{n_j}{n_j+1}, \frac{1+\eta n_j(n_j-2)}{\eta n_j(n_j-1)}\Big\} = d_j\,,$$

where

$$\lambda_j^+ = \Big\{(n_j+1) - \sqrt{(n_j+1)^2 - 4\eta n_j}\Big\}\Big/2n_j\,; \quad j=1,2\,.$$

Now combining Cases I–III, it is easy to define the functions $\mathbf{c}^*$ and $\mathbf{c}_*$ as in (2.1) and (2.2) respectively. The loss function is (1.1), which is the sum of the squared errors, and it is convex with respect to both c_1 and c_1. Then by applying the orbit-by-orbit improvement technique of Brewster and Zidek [2] we get the improved estimators for $\underline{\delta}_{\mathbf{c}}$ in the class D, if either c_1 lies outside the interval $[a_1, b_1]$ (when $\eta \geq 1$) and $[{c_1}^+, d_1]$ (when $0 \leq \eta < 1$) or c_2 lies outside the interval $[a_2, b_2]$ (when $\eta \geq 1$) and $[{c_2}^+, d_2]$ (when $0 \leq \eta < 1$) with probability 1. Applying the Brewster and Zidek [2] technique we have $R(\underline{\delta}_{\mathbf{c}^*}, \underline{\alpha}) \leq R(\underline{\delta}_{\mathbf{c}}, \underline{\alpha})$ when $\eta \geq 1$, and $R(\underline{\delta}_{\mathbf{c}_*}, \underline{\alpha}) \leq R(\underline{\delta}_{\mathbf{c}}, \underline{\alpha})$ when $0 \leq \eta < 1$. This completes the proof of the theorem. $\square$

Proof of Theorem 3.1: The proof of the theorem can be done by using the orbit-by-orbit improvement technique for improving equivariant estimators proposed by Brewster and Zidek [2]. Consider the conditional risk function of $\underline{\delta}_\Psi$ given $W = T_2/T_1$:

$$R\big(\underline{\delta}_{\underline{\Psi}} \,|\, W{=}\,w\big) \;=\; \sum_{i=1}^{2} \frac{1}{\sigma_i^2}\, E\big(Z + T_1\Psi_i(W) - \theta_i\big)^2 \,. \tag{A.4}$$

It is easy to observe that the above risk (A.4) is a convex function of both Ψ_1 and Ψ_2. Hence, the minimizing choice of $\Psi_i(w)$ is obtained as

$$\hat{\Psi}_i(w) \;=\; -\frac{E(Z-\theta_i)\; E(T_1 \,|\, W{=}\,w)}{E(T_1^2 \,|\, W{=}\,w)}\,, \qquad i = 1, 2\,.$$

Using the joint probability density function of (T_1, T_2), we can easily derive the joint probability density function of (T_1, W). The conditional probability density function of T_1 given W is a gamma distribution with shape parameter $n_1 + n_2 - 1$ and scale parameter $1/M$ where $M = \frac{n_1}{\sigma_1} + \frac{n_2}{\sigma_2}\, w$. Hence the conditional expectations are calculated as

$$E(T_1 \,|\, W) \;=\; \frac{n_1 + n_2 - 1}{M}\,, \qquad E(T_1^2 \,|\, W) \;=\; \frac{(n_1+n_2-1)\,(n_1+n_2)}{M^2}\,.$$

Substituting all these values and simplifying we obtain the minimizing choice of $\hat{\Psi}_1$ and $\hat{\Psi}_2$ as

$$\hat{\Psi}_1(w, \tau) \;=\; \frac{\big[\eta - (n_2\tau + n_1)^{-1}\big]\,\big[n_1 + w n_2 \tau\big]}{(n_1+n_2)}$$

and

$$\hat{\Psi}_2(w, \tau) \;=\; \frac{\big[\frac{\eta}{\tau} - (n_2\tau + n_1)^{-1}\big]\,\big[n_1 + w n_2 \tau\big]}{(n_1+n_2)}$$

respectively, where we denote $\tau = \sigma_1/\sigma_2 > 0$.

In order to apply the orbit-by-orbit improvement technique of Brewster and Zidek [2] for improving equivariant estimator, we need the supremum and infimum of both $\hat{\Psi}_1$ and $\hat{\Psi}_2$ with respect to $\tau > 0$ for fixed values of n_1, n_2, η and for given w. We consider the following three separate cases for calculating the supremum and infimum.

Case I: Let $0 < \eta < \min\big\{\frac{1}{n_1}, \frac{1}{n_2}\big\}$. Consider the first component $\hat{\Psi}_1(w, \tau)$. Differentiating with respect to τ we have $\frac{d\hat{\Psi}_1}{d\tau} = \frac{\eta n_2^3 w\tau^2 + 2\eta n_1 n_2^2 w\tau + n_1 n_2(\eta n_1 w - w + 1)}{(n_1+n_2)(n_1+n_2\tau)^2}$. Let $h(\tau) = \eta n_2^3 w\tau^2 + 2\,\eta n_1 n_2^2 w\tau + n_1 n_2(\eta n_1 w - w + 1)$. Now $h(\tau)$ is a convex function of $\tau \in (0, \infty)$. Its minimum is attained at $\tau = -\frac{n_1}{n_2} < 0$. Hence in the region $(0, \infty)$ the minimum will be attained at $\tau = 0$ and the minimum value of $h(\tau)$ is $n_1 n_2(1 - w + \eta n_1 w)$. Assume that the minimum value is positive that is $0 < w \leq \frac{1}{1-\eta n_1}$. For this case $h(\tau) \geq 0$ for $\tau \in (0, \infty)$. Hence the function $\hat{\Psi}_1(w, \tau)$ is an increasing function of $\tau > o$. Hence we have

$$\inf_{\tau>0} \hat{\Psi}_1(w,\tau) = \frac{\eta n_1 - 1}{n_1 + n_2} \quad \text{and} \quad \sup_{\tau>0} \hat{\Psi}_1(w,\tau) = \infty, \quad \text{when} \quad 0 < w \leq \frac{1}{(1-\eta n_1)} .$$

If $w > \frac{1}{1-\eta n_1}$, then the minimum value of $h(\tau)$ is negative and it will cross the τ axis. The function $h(\tau)$ has two real roots say $\tau^- = -\frac{n_1}{n_2} - \frac{1}{n_2}\sqrt{\frac{n_1(w-1)}{\eta w}}$ and $\tau^+ = -\frac{n_1}{n_2} + \frac{1}{n_2}\sqrt{\frac{n_1(w-1)}{\eta w}}$. It is easy to observe that $\tau^- < 0$ and $\tau^+ > 0$. Hence $h(\tau) < 0$ in the region $0 < \tau < \tau^+$ and $h(\tau) \geq 0$ in the region $\tau^+ < \tau < \infty$. Hence the function $\hat{\Psi}_1(w, \tau)$ is decreasing in the region $0 < \tau < \tau^+$ and increasing in the region $\tau^+ < \tau < \infty$. Hence we have

$$\inf_{\tau>0} \hat{\Psi}_1(w,\tau) = \hat{\Psi}_1(w,\tau^+) \quad \text{and} \quad \sup_{\tau>0} \hat{\Psi}_1(w,\tau) = \infty, \quad \text{when} \quad w > \frac{1}{(1-\eta n_1)} ,$$

where

$$\hat{\Psi}_1(w,\tau^+) = \frac{\left[\eta - (n_2\tau^+ + n_1)^{-1}\right]\left[n_1 + w n_2 \tau^+\right]}{(n_1+n_2)} .$$

Next consider the second component $\hat{\Psi}_2$. The derivative of $\hat{\Psi}_2$ with respect to τ is $g(\tau) = \tau^2(n_1 n_2 - \eta n_1 n_2^2 - n_1 n_2 w) - 2\,\eta n_1^2 n_2 \tau - \eta n_1^3$ multiplied by a positive factor. For this case $g(\tau)$ is a convex function of $\tau > 0$. The minimum attained at $\tau = \frac{\eta n_1}{1-\eta n_2 - w} > 0$. Its minimum value is $\frac{\eta n_1^3(w-1)}{1-\eta n_2-w} < 0$ as $w < 1$. Since the minimum value of $g(\tau)$ is negative, it will cross the τ axis. The equation $g(\tau) = 0$ has two real roots say $\alpha^- = \frac{\eta n_1}{1-\eta n_2-w} - \frac{n_1}{n_2}\frac{\sqrt{\eta n_2(1-w)}}{1-\eta n_2-w}$ and $\alpha^+ = \frac{\eta n_1}{1-\eta n_2-w} + \frac{n_1}{n_2}\frac{\sqrt{\eta n_2(1-w)}}{1-\eta n_2-w}$. It is noticed that $\alpha^- < 0$ and $0 < \frac{\eta n_1}{1-\eta n_2-w} < \alpha^+$. Hence the function $g(\tau) < 0$ in the region $(0, \alpha^+)$ and $g(\tau) \geq 0$ in the region $[\alpha^+, \infty)$. This implies that $\hat{\Psi}_2(w, \tau)$ is decreasing in the region $(0, \alpha^+)$ and increasing in the region $[\alpha^+, \infty)$. Hence we have

$$\inf_{\tau>0} \hat{\Psi}_2(w,\tau) = \hat{\Psi}_2(w,\alpha^+) \quad \text{and} \quad \sup_{\tau>0} \hat{\Psi}_2(w,\tau) = \max\left\{\hat{\Psi}_2(w,0), \hat{\Psi}_2(w,\infty)\right\} = \infty ,$$

where

$$\hat{\Psi}_2(w,\alpha^+) = \frac{1}{n_1+n_2}\left[\frac{\eta}{\alpha^+} - \frac{1}{n_1 + n_2\alpha^+}\right]\left[n_1 + n_2 w \alpha^+\right],$$

when $1 - \eta n_2 \geq w$.

Now assume that $1 - \eta n_2 < w$. Then the function $g(\tau)$ is a concave function of τ. Its maximum value is attained at $\tau = \frac{\eta n_1}{1-\eta n_2-w} < 0$. Hence within the concerned region the maximum is attained at $\tau = 0$. Its maximum value is $-\eta n_1^3 < 0$. This implies that the function $g(\tau) < 0$ in the region $(0, \infty)$. Thus the function $\hat{\Psi}_2(w, \tau)$ is decreasing in $\tau \in (0, \infty)$. Hence we have

$$\inf_{\tau>0} \hat{\Psi}_2(w,\tau) = \frac{n_2 w}{n_1+n_2}\left(\eta - \frac{1}{n_2}\right) \quad \text{and} \quad \sup_{\tau>0} \hat{\Psi}_2(w,\tau) = \infty ,$$

when $1 - \eta n_2 < w$.

Case II: Let $\eta \geq \max\{\frac{1}{n_1}, \frac{1}{n_2}\}$. Consider the first component $\Psi_1(w,\tau)$. Now the derivative of $\hat{\Psi}_1(w,\tau)$ with respect to τ is $h(\tau)$ multiplied by a positive factor. As in Case I, the function $h(\tau)$ is a convex function of τ. The minimum is attained at $\tau = -\frac{n_1}{n_2} < 0$. Hence within the interval $(0,\infty)$ the minimum is attained at $\tau = 0$. Its minimum value is $n_1 n_2(1 - w + \eta n_1 w) \geq 0$ as $\eta \geq \max\{\frac{1}{n_1}, \frac{1}{n_2}\}$. Hence $h(\tau) \geq 0$, $\forall\, \tau > 0$. Thus the function $\hat{\Psi}_1(w,\tau)$ is increasing in the region $(0,\infty)$. Thus we have

$$\inf_{\tau>0} \hat{\Psi}_1(w,\tau) = \hat{\Psi}_1(w,0) = \frac{\eta n_1 - 1}{n_1 + n_2} \quad \text{and} \quad \sup_{\tau>0} \hat{\Psi}_1(w,\tau) = \hat{\Psi}_1(w,\infty) = \infty\,.$$

Consider the second component $\hat{\Psi}_2(w,\tau)$. As in Case I, the derivative of $\hat{\Psi}_2(w,\tau)$ is simply $g(\tau)$ multiplied by a positive factor. Also under the condition $\eta \geq \max\{\frac{1}{n_1}, \frac{1}{n_2}\}$, the only possibility is $1 - \eta n_2 < w$. The function $g(\tau)$ is a concave function and the maximum is attained at $\tau = \frac{\eta n_1}{1 - \eta n_2 - w} < 0$. Hence the maximum will be attained at $\tau = 0$ in the concerned region $(0,\infty)$. The maximum value is $-\eta n_1^3 < 0$. Hence $g(\tau) < 0$, $\forall\, \tau > 0$. Thus the function $\hat{\Psi}_2(w,\tau)$ is decreasing in the region $(0,\infty)$. Thus we have

$$\inf_{\tau>0} \hat{\Psi}_2(w,\tau) = \frac{n_2 w}{n_1 + n_2}\left(\eta - \frac{1}{n_2}\right) \quad \text{and} \quad \sup_{\tau>0} \hat{\Psi}_2(w,\tau) = \infty\,.$$

Case III: Let $\min\{\frac{1}{n_1}, \frac{1}{n_2}\} \leq \eta < \max\{\frac{1}{n_1}, \frac{1}{n_2}\}$. For this case we have two possibilities either $\frac{1}{n_1} \leq \eta < \frac{1}{n_2}$ or $\frac{1}{n_2} \leq \eta < \frac{1}{n_1}$. Analyzing as in the above cases we have for $\frac{1}{n_1} \leq \eta < \frac{1}{n_2}$,

$$\inf_{\tau>0} \hat{\Psi}_1(w,\tau) = \hat{\Psi}_1(w,0) = \frac{\eta n_1 - 1}{n_1 + n_2} \quad \text{and} \quad \sup_{\tau>0} \hat{\Psi}_1(w,\tau) = \hat{\Psi}_1(w,\infty) = \infty\,,$$

and

$$\inf_{\tau>0} \hat{\Psi}_2(w,\tau) = \begin{cases} \hat{\Psi}_2(w,\alpha^+)\,, & \text{if } w \leq 1 - \eta n_2\,, \\ \frac{n_2 w}{n_1+n_2}\left(\eta - \frac{1}{n_2}\right), & \text{if } w > 1 - \eta n_2\,, \end{cases} \quad \text{and} \quad \sup_{\tau>0} \hat{\Psi}_2(w,\tau) = +\infty\,.$$

Likewise when $\frac{1}{n_2} \leq \eta < \frac{1}{n_1}$, we have

$$\inf_{\tau>0} \hat{\Psi}_1(w,\tau) = \begin{cases} \frac{\eta n_1 - 1}{n_1+n_2}\,, & \text{if } w \leq \frac{1}{1-\eta n_1}\,, \\ \hat{\Psi}_1(w,\tau^+), & \text{if } w > \frac{1}{1-\eta n_1}\,, \end{cases} \quad \text{and} \quad \sup_{\tau>0} \hat{\Psi}_1(w,\tau) = +\infty\,,$$

and

$$\inf_{\tau>0} \hat{\Psi}_2(w,\tau) = \frac{n_2 w}{n_1 + n_2}\left(\eta - \frac{1}{n_2}\right) \quad \text{and} \quad \sup_{\tau>0} \hat{\Psi}_2(w,\tau) = \infty\,.$$

Now it is easy to define the functions $\underline{\Psi}_0$ $\big($when $0<\eta\le\min\{\frac{1}{n_1},\frac{1}{n_2}\}\big)$, $\underline{\Psi}^0$ $\big($when $\eta\ge\max\{\frac{1}{n_1},\frac{1}{n_2}\}\big)$, $\underline{\Psi}_{11}$ $\big($when $\frac{1}{n_1}\le\eta<\frac{1}{n_2}\big)$ and $\underline{\Psi}_{22}$ $\big($when $\frac{1}{n_2}\le\eta<\frac{1}{n_1}\big)$ as defined in (3.4), (3.5), (3.6) and (3.7) respectively. Since the loss function (1.2) is a sum of the convex loss functions with respect to both $\hat{\Psi}_1$ and $\hat{\Psi}_2$, an application of the Theorem 3.3.1 of Brewster and Zidek [2], gives $R(\underline{\delta}_{\underline{\Psi}_0},\underline{\alpha})\le R(\underline{\delta}_{\underline{\Psi}},\underline{\alpha})$ if there exist some values of parameters $\underline{\alpha}$ such that $P_{\underline{\alpha}}(\underline{\Psi}_0\ne\underline{\Psi})>0$ for the case $0<\eta\le\min\{\frac{1}{n_1},\frac{1}{n_2}\}$. Similarly by applying the Brewster and Zidek [2] technique for the case $\eta\ge\max\{\frac{1}{n_1},\frac{1}{n_2}\}$, we have $R(\underline{\delta}_{\underline{\Psi}^0},\underline{\alpha})\le R(\underline{\delta}_{\underline{\Psi}},\underline{\alpha})$ if there exist some values of parameters $\underline{\alpha}$ such that $P_{\underline{\alpha}}(\underline{\Psi}^0\ne\underline{\Psi})>0$. When $\frac{1}{n_1}\le\eta<\frac{1}{n_2}$ the estimator $\underline{\delta}_{\underline{\Psi}_{11}}$ improves upon $\underline{\delta}_{\underline{\Psi}}$ if $P_{\underline{\alpha}}(\underline{\Psi}_{11}\ne\underline{\Psi})>0$ for some choices of $\underline{\alpha}$. When $\frac{1}{n_2}\le\eta<\frac{1}{n_1}$ the estimator $\underline{\delta}_{\underline{\Psi}_{22}}$ improves upon $\underline{\delta}_{\underline{\Psi}}$ if $P_{\underline{\alpha}}(\underline{\Psi}_{22}\ne\underline{\Psi})>0$ for some choices of $\underline{\alpha}$. This completes the proof of the theorem. $\square$

ACKNOWLEDGMENTS

The first author (Manas Ranjan Tripathy) would like to thank DST (SERB), grant number SR/FTP/MS-37/2012 dated 04.10.2013 New Delhi, India for providing the financial support. The authors would like to thank the two unanimous referees and the editors for their valuable suggestions and comments which helped in strengthening the article.

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