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BOOTSTRAP PREDICTION INTERVAL FOR ARMA MODELS WITH UNKNOWN ORDERS

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Abstract:

- This paper aims to investigate the construction of the prediction intervals for ARMA (p, q) models with unknown orders. We present the bootstrap algorithms for the prediction intervals based on the bootstrap distribution of orders (p, q). The asymptotic properties of the intervals are also discussed. The Monte Carlo simulation studies show that the proposed algorithm significantly improves the coverage accuracy of the prediction interval compared to the methods using pre-estimated values of orders, especially when the sample size is small and the true model order is low.

Key-Words:

- *ARMA model; asymptotic properties; bootstrap; prediction interval.*

AMS Subject Classification:

- 62M10, 62F40.

1. INTRODUCTION

As indicated in [13], the construction of prediction intervals for time series models is both important and intriguing, see also, e.g., [1, 2, 4, 6, 7], [10], [12] and [15, 17, 18]. The prediction intervals can show explicitly the uncertainty underlying the estimation procedure and measure the accuracy with the point predictors.

An early review article by [12] covered several methods of construction of prediction intervals for linear regression models under normal assumption. [15] proposed an asymptotically valid prediction interval for linear models based on normal approximations. It is known that Gaussian-based prediction intervals produce poor coverages when the distributional assumptions are violated. As a remedy, some form of resampling, for example the residual-based bootstrap, is necessary.

The literature on predictive intervals for time series is not large. In order to construct prediction intervals without the assumption of Gaussianity, [11] developed a coherent bootstrap algorithm of constructing prediction intervals for time series that can be modeled as linear, nonlinear or nonparametric autoregression AR(p) with known order p. Their bootstrap intervals are able to capture the predictor variability due to the innovation errors as well as the estimation errors.

For other time series models, [3] presented a bootstrap approach called The Boot.EXPOS to forecast time series through combining the use of exponential smoothing methods with the bootstrap methodology. [14] developed a bootstrap prediction interval procedure by using a pre-estimated order of the AR approximation for FARIMA processes. [9] provided a bootstrap method for estimating the parameters of ARMA (p, q) models. [13] derived model-free prediction intervals based on a new model-free prediction principle and bootstrapping, which can be applied to nonparametric time series models with known orders.

To the best of our knowledge, the construction of bootstrap prediction intervals based on the bootstrap distribution of the orders for ARMA (p, q) models with unknown orders remain essentially unexplored. This is the issue we intend to address in the current paper.

Section 2 presents the algorithms for the construction of bootstrap prediction intervals for ARMA (p, q) models with known orders and unknown orders, respectively. The asymptotic validity and asymptotic pertinence of the intervals are addressed in Section 3. The paper is concluded with simulation studies comparing the finite sample performance of the proposed method with other methods in terms of coverage level and length of interval in Section 4. The proofs of the theorems are given in the Appendix.

2. BOOTSTRAP PREDICTION INTERVALS FOR ARMA MODELS

Consider the strictly stationary, causal ARMA (p, q) model defined by the recursion

$$X_t = \sum_{j=1}^{p} a_j X_{t-j} + \sum_{j=0}^{q} b_j \varepsilon_{t-j}, \qquad t \in \mathbb{Z}, \quad b_0 = 1, \tag{2.1}$$

with $\{\varepsilon_t\}$ being i.i.d. with mean zero, variance σ^2, where $1-\sum_{j=1}^{p} a_j z^j \neq 0$ and $\sum_{j=0}^{q} b_j z^j \neq 0$ for $|z| \leq 1$, and $1-\sum_{j=1}^{p} a_j z^j$ and $\sum_{j=0}^{q} b_j z^j$ have no zeros in common.

Suppose that we have the observations $\{X_t,\ t=1,2,...,n\}$ and denote by $\hat{X}_{n+1}$ the point predictor of X_{n+1} based on the data $X_1,...,X_n$. Let $\theta = (a_1,...,a_p,b_1,...,b_q)^{\mathsf{T}}$. [8] defined an M-estimator $\hat{\theta}^M$ of θ for model (2.1), where $\hat{\theta}^M = (\hat{a}^M_{1,n},...,\hat{a}^M_{p,n},\hat{b}^M_{1,n},...,\hat{b}^M_{q,n})^{\mathsf{T}}$ is the solution of the equation

$$\Psi_n(\theta) \;=\; \frac{1}{\sqrt{n}}\sum_{j=1}^{n}\psi\big(\varepsilon_j(\theta)\big)\,Z(j-1;\theta) \;=\; 0$$

for some suitably chosen score function ψ, where

$$\varepsilon_j(\theta) \;=\; \sum_{k=0}^{j-1}\beta_k(\theta)\left(X_{j-k}-\sum_{i=1}^{p}a_i\,X_{j-k-i}\right), \qquad j=1,...,n\,,$$

and

$$Z(j-1;\theta) \;=\; \sum_{k=0}^{j-1}\beta_k(\theta)\Big(X\big(j-1-k\big)^{\mathsf{T}},\,E\big(j-1-k;\,\theta\big)^{\mathsf{T}}\Big)^{\mathsf{T}}, \qquad j=1,...,n\,,$$

where $\beta_k(\theta)$ satisfies

$$\sum_{k=0}^{\infty}\beta_k(\theta)\,z^k \;=\; \left(1+\sum_{j=1}^{q}b_j\,z^j\right)^{-1}, \qquad |z|\leq 1\,,$$

and $X(j-1) = \big(X_{j-1},...,X_{j-p}\big)^{\mathsf{T}}$, $E(j-1;\theta) = \big(\varepsilon_{j-1}(\theta),...,\varepsilon_{j-q}(\theta)\big)^{\mathsf{T}}$, $X_j = 0$ and $\varepsilon_j(\theta) = 0$ for $j \leq 0$.

Theorem 3.1 of [8] showed that, under some mild conditions, $\sqrt{n}\,\big(\hat{\theta}^M-\theta\big)$ is asymptotically normally distributed. Therefore, we define $\hat{X}_{n+1}$ as

$$\hat{X}_{n+1} \;=\; \sum_{k=1}^{p}\hat{a}^M_{k,n}\,X_{n+1-k} + \sum_{k=1}^{q}\hat{b}^M_{k,n}\,\hat{\varepsilon}_{n+1-k,n}\big(\hat{\theta}^M\big)\,, \tag{2.2}$$

where

$$\hat{\varepsilon}_{j,n}\big(\hat{\theta}^M\big) \;=\; \sum_{k=0}^{j-1}\beta_k\big(\hat{\theta}^M\big)\left(X_{j-k}-\sum_{i=1}^{p}\hat{a}^M_{i,n}\,X_{j-k-i}\right), \qquad j=1,...,n\,. \tag{2.3}$$

2.1. Bootstrap prediction intervals for ARMA models with known orders

In this subsection we assume that the orders p and q of the ARMA (p,q) model are known. Inspired by the algorithms discussed in [11], we provide the following bootstrap algorithm of the prediction interval for X_{n+1} based on (2.2). Let the one-step ahead predictive root be defined as $X_{n+1}-\hat{X}_{n+1}$. The algorithm actually uses the distribution of the bootstrap predictive root to estimate the distribution of the true predictive root.

B-ARMARoots Algorithm:

1. Use observations $X_1, ..., X_n$ to obtain M-estimator $\hat{\theta}^M$ for model (2.1).
2. Compute the fitted residuals $\hat{\varepsilon}_{j,n}\big(\hat{\theta}^M\big)$ from (2.3), denote by $\hat{\varepsilon}_{\cdot,n}$ the mean of the fitted residuals, center the fitted residuals, and compute the empirical distribution $\hat{F}_n$ of $\hat{\varepsilon}_{j,n} - \hat{\varepsilon}_{\cdot,n}$

$$\hat{F}_n(x) = \frac{1}{n}\sum_{j=1}^{n} \mathbf{1}_{[\hat{\varepsilon}_{j,n}-\hat{\varepsilon}_{\cdot,n},\infty)}(x)\,, \qquad x \in \mathbb{R}\,,$$

   where $\mathbf{1}_A$ is the indicator function of set A.
3. Compute the predicted future value $\hat{X}_{n+1}$ using (2.2).
4. Draw bootstrap pseudo residuals $\{\varepsilon_j^*\}$ from $\hat{F}_n$, calculate the pseudo-data $\{X_j^*\}$ by

$$X_j^* = \sum_{k=1}^{p} \hat{a}_{k,n}^M X_{j-k}^* + \sum_{k=1}^{q} \hat{b}_{k,n}^M \varepsilon_{j-k}^* + \varepsilon_j^*\,.$$

5. Use the pseudo-data $X_1^*, ..., X_n^*$ to obtain M-estimator $\hat{\theta}^{M,*}$. Then compute the residuals

$$\hat{\varepsilon}_{j,n}^*\big(\hat{\theta}^{M,*}\big) = \sum_{k=0}^{j-1} \beta_k\big(\hat{\theta}^{M,*}\big)\left(X_{j-k} - \sum_{i=1}^{p} \hat{a}_{i,n}^{M,*} X_{j-k-i}\right), \qquad j = 1, ..., n\,.$$

6. Compute the bootstrap predicted future value

$$\hat{X}_{n+1}^* = \sum_{k=1}^{p} \hat{a}_{k,n}^{M,*} X_{n+1-k} + \sum_{k=1}^{q} \hat{b}_{k,n}^{M,*}\, \hat{\varepsilon}_{n+1-k,n}^*\big(\hat{\theta}^{M,*}\big)\,.$$

7. Compute the future bootstrap observation X_{n+1}^*

$$X_{n+1}^* = \sum_{k=1}^{p} \hat{a}_{k,n}^M X_{n+1-k} + \sum_{k=1}^{q} \hat{b}_{k,n}^M\, \hat{\varepsilon}_{n+1-k,n}\big(\hat{\theta}^M\big) + \varepsilon_{n+1}^*\,.$$

   Then compute the one-step ahead predictive root $X_{n+1}^* - \hat{X}_{n+1}^*$.
8. Repeat Steps 4–7 above B times, compute the empirical distribution of $X_{n+1}^* - \hat{X}_{n+1}^*$ whose $\alpha/2$-quantile is denoted by $q(\alpha/2)$. Construct the $(1-\alpha)\,100\%$ equal-tailed prediction interval for X_{n+1} as

$$\Big[\hat{X}_{n+1} + q(\alpha/2)\,,\ \hat{X}_{n+1} + q(1-\alpha/2)\Big]\,. \tag{2.4}$$

In Step 4, to ensure the stationarity of the bootstrap series, one usually generates $n+m$ pseudo residuals from $\hat{F}_n$ for some large positive m to compute the pseudo-data $\{X_j^*\}$, and then discard the first m data.

2.2. Bootstrap prediction intervals for ARMA models with unknown orders

In practice, the orders of ARMA models are usually unknown. In this subsection we introduce the bootstrap prediction interval under this case.

Given observations $X_1, ..., X_n$ from ARMA (p, q) model (2.1), to construct the prediction interval for X_{n+1}, an intuitive method one may consider is the following

OB-ARMARoots Algorithm:

1. Determine the orders p_0, q_0 using, e.g., AIC criteria.

2. Apply B-ARMARoots Algorithm to construct the prediction interval for ARMA (p_0, q_0) model.

Since AIC is biased when the sample size is small, [5] presented a bootstrap version of AIC, denoted AIC*, which generally outperforms the original AIC. AIC* is obtained by bootstrapping both the likelihood and the bias term of AIC. That is,

$$\mathrm{AIC}^* = -2\,\log\bigl(L(\hat{\theta}^*|x^*)\bigr) + (2\,K)^*, \qquad K = p + q\,,$$

with K the number of parameters and $L(\cdot)$ the log likelihood function. In the case of ARMA models, $L(\cdot)$ can be computed by using the estimated variance of the residuals.

Applying the procedure in [5], we can obtain the bootstrap distribution of the orders (p, q). Next we propose the bootstrap algorithm of the prediction interval based on the bootstrap distribution of the orders.

CB-ARMARoots Algorithm:

1. Determine a pair of maximum orders for ARMA model, say (P, Q).

2. Approximate an $\mathrm{AR}\bigl(p(n)\bigr)$ model to $X_1, ..., X_n$. The order $p(n)$ can be selected by using the AIC criteria. Then construct the estimators of the autoregressive coefficients $\hat{\varphi}_{1,n}, ..., \hat{\varphi}_{p(n),n}$ using Yule–Walker method, compute the residuals

$$\hat{\varepsilon}_{t,n} = X_t - \sum_{j=1}^{p(n)} \hat{\varphi}_{j,n}\, X_{t-j}\,.$$

3. Center the residuals, $\tilde{\varepsilon}_{t,n} = \hat{\varepsilon}_{t,n} - \bigl(n - p(n)\bigr)^{-1} \sum_{j=p(n)+1}^{n} \hat{\varepsilon}_{j,n}$, and compute the empirical distribution of $\{\tilde{\varepsilon}_{t,n}\}$

$$\hat{F}_{\varepsilon,n}(x) = \bigl(n - p(n)\bigr)^{-1} \sum_{t=p(n)+1}^{n} \mathbf{1}_{[\tilde{\varepsilon}_{t,n} \le x]}\,.$$

4. Draw bootstrap pseudo residuals $\{\varepsilon_t^*\}$ from $\hat{F}_{\varepsilon,n}$, generate bootstrap sample $\{X_t^*\}$ by the recursion

$$X_t^* = \sum_{j=1}^{p(n)} \hat{\varphi}_{j,n}\, X_{t-j}^* + \varepsilon_t^*\,.$$

5. For bootstrap sample $\{X_t^*\}$, fit an exhaustive set of size $(P+1)*(Q+1)$ of tentative ARMA (p, q), compute AIC* for all the candidate pairs (p, q). Then $(\hat{p}^*, \hat{q}^*)$ is identified according to

$$(\hat{p}^*, \hat{q}^*) = \arg \min_{p \le P,\, q \le Q} \mathrm{AIC}^*(p, q)\,.$$

6. Repeat Steps 4–5 B_1 times, denote the number of the pairs (p,q) out of the group of the B_1 pairs as $b_{p,q}$.
7. For any pair (p,q) in Step 6, Steps 1–7 in B-ARMARoots Algorithm are repeated $B_2\times b_{p,q}$ times to obtain $B_2\times b_{p,q}$ values of $X^*_{n+1}-\hat{X}^*_{n+1}$.
8. Repeat Step 7 to obtain $B_1\times B_2$ values of $X^*_{n+1}-\hat{X}^*_{n+1}$. Then compute the empirical distribution of $X^*_{n+1}-\hat{X}^*_{n+1}$ whose $\alpha/2$-quantile is denoted by $q(\alpha/2)$. Construct the $(1-\alpha)100\%$ equal-tailed prediction interval for X_{n+1} as

$$\Big[\hat{X}_{n+1}+q(\alpha/2)\,,\;\hat{X}_{n+1}+q(1-\alpha/2)\Big]\,. \tag{2.5}$$

In Step 1, the choice of the maximum order (P,Q) is *a priori* and arbitrary. For computational reasons and from a practical point of view, (P,Q) should not be too large. For example, economic time series can usually be modeled as ARMA processes with orders not greater than 3. Therefore, in the simulation studies in Section 4, we set $P=Q=5$.

In Step 2, we use the AIC criteria to determine the order $p(n)$, because, as discussed in [16], the order selected from the AIC criteria is asymptotically efficient for the infinite order autoregressive models. In the simulation studies we select $p(n)$ that minimizes the AIC evaluated over a range of $[1,10\log_{10}(n)]$. Moreover, we can also select the order $p(n)$ by iterative estimate of the spectral density on the residuals coming from the fitting procedure of tentative autoregressive models until closeness to a constant is reached. That is, starting from $\tilde{p}=1$, we fit an AR$(\tilde{p})$ model to $X_1,\ldots,X_n$ using e.g. Yule–Walker method and then estimate the spectral density of the residuals. If the estimated spectral density is close to a constant, stop and set $p(n)=\tilde{p}$. Otherwise, let $\tilde{p}=\tilde{p}+1$, repeat the previous procedure until the estimator of the spectral density is almost a constant.

In Step 6, $b_{p,q}$ stands for the number of (p,q) appeared among all the B_1 pairs obtained by Step 5. For example, let $B_1=1000$, after Steps 4–5 are repeated 1000 times, we obtain 1000 pairs of orders $(\hat{p}^*,\hat{q}^*)$ from Step 5. If, among all these pairs, $(1,1)$ appears 10 times, then $b_{1,1}=10$.

Again in Step 4, to ensure the stationarity of the bootstrap series, we use the techniques mentioned at the end of Section 2.1.

3. ASYMPTOTIC PROPERTIES OF BOOTSTRAP PREDICTION INTERVALS

In this section we investigate the asymptotic properties of bootstrap prediction intervals proposed in the previous section. Using the notations in [11], we define the asymptotic validity and the asymptotic pertinence of bootstrap prediction intervals for ARMA (p,q) models.

Definition 3.1 (Asymptotic validity of bootstrap prediction interval)**.** Let L_n, U_n be the functions of $X_1,\ldots,X_n$. The interval $[L_n,U_n]$ is called a $(1-\alpha)100\%$ asymptotically valid prediction interval for X_{n+1} if

$$P\big(L_n\le X_{n+1}\le U_n\big)\to 1-\alpha \qquad \text{as}\quad n\to\infty\,.$$

Let

$$
\begin{aligned}
A_n &= \sum_{j=1}^{p} a_j\, X_{n+1-j} - \sum_{j=1}^{p} \hat{a}_{j,n}^{M}\, X_{n+1-j}\,,\\
B_n &= \sum_{j=1}^{q} b_j\, \varepsilon_{n+1-j} - \sum_{j=1}^{q} \hat{b}_{j,n}^{M}\, \hat{\varepsilon}_{n+1-j,n}\big(\hat{\theta}^{M}\big)\,,\\
A_n^{*} &= \sum_{j=1}^{p} \hat{a}_{j,n}^{M}\, X_{n+1-j} - \sum_{j=1}^{p} \hat{a}_{j,n}^{M,*}\, X_{n+1-j}
\end{aligned}
$$

and

$$
B_n^{*} = \sum_{j=1}^{q} \hat{b}_{j,n}^{M}\, \hat{\varepsilon}_{n+1-j,n}\big(\hat{\theta}^{M}\big) - \sum_{j=1}^{q} \hat{b}_{j,n}^{M,*}\, \hat{\varepsilon}_{n+1-j,n}^{*}\big(\hat{\theta}^{M,*}\big)\,.
$$

Then, the predictive root and the bootstrap predictive root can be written as

$$
X_{n+1} - \hat{X}_{n+1} = \varepsilon_{n+1} + A_n + B_n
$$

and

$$
X_{n+1}^{*} - \hat{X}_{n+1}^{*} = \varepsilon_{n+1}^{*} + A_n^{*} + B_n^{*}\,.
$$

Definition 3.2 (Asymptotic pertinence of bootstrap prediction interval)**.** A bootstrap prediction interval is called asymptotically pertinent provided the bootstrap satisfies the following four conditions

- **(i)** $\sup_x \big|P\big(\varepsilon_{n+1} \le x\big) - P\big(\varepsilon_{n+1}^{*} \le x\big)\big| \to^{p} 0$ as $n \to \infty$, where $\to^{p}$ stands for convergence in probability.
- **(ii)** $A_n + B_n \to^{p} 0$ and $A_n^{*} + B_n^{*} \to^{p} 0$ as $n \to \infty$.
- **(iii)** $\big|P\big(a_n A_n \le x\big) - P\big(a_n A_n^{*} \le x\big)\big| \to^{p} 0$ for some sequence $a_n \to \infty$, for all points x where $P\big(a_n A_n \le x\big)$ is continuous.
- **(iv)** ε_{n+1}^{*} and A_n^{*} are independent.

The following two theorems address the asymptotic validity and the stronger property of asymptotic pertinence of the prediction interval (2.4) using B-ARMA Roots Algorithm.

Theorem 3.1. *For ARMA model (2.1) with known orders p and q, the prediction interval (2.4) is asymptotically valid.*

Theorem 3.2. *For ARMA model (2.1) with known orders p and q, the prediction interval (2.4) is asymptotically pertinent.*

The next theorem gives the asymptotic validity and the asymptotic pertinence of the prediction interval (2.5) using CB-ARMARoots Algorithm.

Theorem 3.3. *For ARMA model (2.1) with unknown orders p an q, the prediction interval (2.5) is asymptotically valid and asymptotically pertinent.*

4. MONTE CARLO SIMULATIONS

In this section, we evaluate and compare the performance of the proposed CB-ARMA algorithm based on the bootstrap distribution of the orders and the OB-ARMA algorithm with pre-estimated fixed orders, as well as two types of bootstrap prediction intervals via predictive roots and percentile methods, respectively.

In contrast to the predictive root methods adopted in this paper, the percentile method uses the bootstrap distribution of X^*_{n+1} to estimate the distribution of the future value X_{n+1} instead of using the distribution of the bootstrap predictive root to estimate the distribution of the true predictive root, where

$$X^*_{n+1} = \sum_{k=1}^{p} \hat{a}^{M,*}_{k,n} X_{n+1-k} + \sum_{k=1}^{q} \hat{b}^{M,*}_{k,n} \hat{\varepsilon}^*_{n+1-k,n}\big(\hat{\theta}^{M,*}\big) + \varepsilon^*_{n+1} .$$

Data are generated from the following eight models with sample sizes $n = 25, 50, 75, 100, 200$ and 400 for each model.

(**1**) ARMA(1,1) model:

$$X_{t+1} = 0.5\, X_t + 0.5\, \varepsilon_t + \varepsilon_{t+1} ,$$

where errors $\{\varepsilon_t\}$ are from N(0,1);

(**2**) ARMA(1,1) model:

$$X_{t+1} = 0.5\, X_t + 0.5\, \varepsilon_t + \varepsilon_{t+1} ,$$

where errors $\{\varepsilon_t\}$ are from Laplace distribution with mean 0 and variance 1;

(**3**) ARMA(1,2) model:

$$X_{t+1} = 0.5\, X_t + 0.4\, \varepsilon_t + 0.2\, \varepsilon_{t-1} + \varepsilon_{t+1} ,$$

where errors $\{\varepsilon_t\}$ are from N(0,1);

(**4**) ARMA(1,2) model:

$$X_{t+1} = 0.5\, X_t + 0.4\, \varepsilon_t + 0.2\, \varepsilon_{t-1} + \varepsilon_{t+1} ,$$

where errors $\{\varepsilon_t\}$ are from Laplace distribution with mean 0 and variance 1;

(**5**) ARMA(2,2) model:

$$X_{t+1} = -0.5\, X_t + 0.4\, X_{t-1} + 0.4\, \varepsilon_t + 0.2\, \varepsilon_{t-1} + \varepsilon_{t+1} ,$$

where errors $\{\varepsilon_t\}$ are from N(0,1);

(**6**) ARMA(2,2) model:

$$X_{t+1} = -0.5\, X_t + 0.4\, X_{t-1} + 0.4\, \varepsilon_t + 0.2\, \varepsilon_{t-1} + \varepsilon_{t+1} ,$$

where errors $\{\varepsilon_t\}$ are from Laplace distribution with mean 0 and variance 1.

(**7**) ARMA(3,1) model:

$$X_{t+1} = -0.5\,X_t + 0.2\,X_{t-1} + 0.2\,X_{t-2} + 0.4\,\varepsilon_t + \varepsilon_{t+1}\,,$$

where errors $\{\varepsilon_t\}$ are from N(0,1);

(**8**) ARMA(3,1) model:

$$X_{t+1} = -0.5\,X_t + 0.2\,X_{t-1} + 0.2\,X_{t-2} + 0.4\,\varepsilon_t + \varepsilon_{t+1}\,,$$

where errors $\{\varepsilon_t\}$ are from Laplace distribution with mean 0 and variance 1.

We use four bootstrap methods to create $B = 1000$ bootstrap pseudo-series, respectively, and construct the prediction intervals $[L, U]$ with nominal coverage levels of 95% and 90%. For CB-ARMA algorithm, we choose $B_1 = B_2 = 1000$. To assess the corresponding empirical coverage level (CVR) and average length (LEN) of the constructed interval, we also generate 1000 one-step ahead future values $X_{n+1,j} = \sum_{k=1}^{p} \hat{a}_{k,n}^{M}\, X_{n+1-k} + \sum_{k=1}^{q} \hat{b}_{k,n}^{M}\, \hat{\varepsilon}_{n+1-k,n}\big(\hat{\theta}^{M}\big) + \varepsilon_j^*$. Then, CVR and LEN are given by

$$\text{CVR} = \frac{1}{1000}\sum_{j=1}^{1000} \mathbf{1}_{[L,U]}\big(X_{n+1,j}\big)\,, \qquad \text{LEN} = U - L\,.$$

Tables 1–8 report the simulation results for eight models using four bootstrap methods. Generally speaking, the CB-ARMA algorithm based on the bootstrap distribution of the orders is generally superior to OB-ARMA algorithm. The bootstrap prediction intervals using CB-ARMARoots method proposed in this paper uniformly improve CVRs as compared to the other three methods, while in most cases the interval length is increased as a price to pay for using CB-ARMA algorithm. But there are some exceptions that the LEN of CB-ARMA intervals is smaller than the LEN of OB-ARMA intervals when $n = 200$ or $n = 400$ in Tables 2, 3, 5 and 8.

Comparing Tables 1–4 with Tables 5–8, we see that, when the true order of the ARMA model is lower, the OB-ARMA algorithm performs worse with respect to the coverage level. Because the AIC criterion tends to select higher order which results in overfitting.

From Tables 1–2 it is clear that, when the sample size is small ($n \le 200$) and the order selected from AIC criterion is larger than the true model order, CB-ARMA algorithm compares favorably with OB-ARMA algorithm. When the sample size is large ($n = 400$), CB-ARMA algorithm still outperforms OB-ARMA algorithm in terms of coverage in most cases, but the improvement is not as big as that for the case of small sample sizes. Table 8 implies that, for large sample size and high model order, CB-ARMA and OB-ARMA have similar coverage level but OB-ARMA is slightly superior to CB-ARMA in some cases.

Moreover, ARMARoots algorithm generally offers improvements in the coverage accuracy comparing to ARMAPercentile algorithm, but using CB-ARMARoots method generally increases the length of the intervals. However, when the errors have Laplace distribution, there are several cases where CB-ARMARoots intervals have smaller length compared to CB-ARMAPercentile method.

Table 1: CVR and LEN for ARMA(1,1) with Normal innovations.

Normal	Nominal coverage 95%		Nominal coverage 90%	
	CVR	LEN	CVR	LEN
$n=25$				
CB-ARMARoots	98.7%	6.51	95.9%	4.99
CB-ARMAPercentile	97.4%	5.65	94.2%	4.77
OB-ARMARoots	91.9%	4.55	84.7%	3.91
OB-ARMAPercentile	92.7%	4.59	88.8%	3.97
$n=50$				
CB-ARMARoots	96.6%	4.47	91.4%	3.62
CB-ARMAPercentile	95.0%	4.37	89.8%	3.54
OB-ARMARoots	90.4%	3.96	88.6%	3.21
OB-ARMAPercentile	90.5%	3.84	88.2%	3.15
$n=75$				
CB-ARMARoots	97.3%	5.61	95.0%	4.75
CB-ARMAPercentile	95.8%	5.23	92.5%	4.27
OB-ARMARoots	92.1%	4.23	90.3%	3.93
OB-ARMAPercentile	92.1%	4.22	90.5%	3.95
$n=100$				
CB-ARMARoots	97.4%	4.45	92.9%	3.62
CB-ARMAPercentile	96.2%	4.31	92.0%	3.61
OB-ARMARoots	93.0%	3.65	90.6%	3.38
OB-ARMAPercentile	93.5%	3.68	90.6%	3.38
$n=200$				
CB-ARMARoots	97.2%	5.59	93.2%	4.55
CB-ARMAPercentile	92.7%	5.39	85.8%	4.42
OB-ARMARoots	93.0%	4.99	85.9%	4.15
OB-ARMAPercentile	86.4%	5.01	76.6%	4.18
$n=400$				
CB-ARMARoots	97.8%	4.62	94.6%	3.86
CB-ARMAPercentile	97.7%	4.60	94.5%	3.86
OB-ARMARoots	97.0%	4.36	93.5%	3.71
OB-ARMAPercentile	96.7%	4.20	93.3%	3.67

Table 2: CVR and LEN for ARMA(1,1) with Laplace innovations.

Laplace	Nominal coverage 95%		Nominal coverage 90%	
	CVR	LEN	CVR	LEN
$n=25$				
CB-ARMARoots	93.4%	3.26	80.8%	2.27
CB-ARMAPercentile	89.0%	3.04	79.0%	2.26
OB-ARMARoots	87.0%	2.89	68.3%	1.73
OB-ARMAPercentile	86.9%	2.89	68.7%	1.72
$n=50$				
CB-ARMARoots	97.5%	5.20	94.7%	4.14
CB-ARMAPercentile	97.1%	5.23	92.0%	3.89
OB-ARMARoots	93.7%	5.14	91.8%	4.12
OB-ARMAPercentile	93.8%	5.13	91.4%	4.09
$n=75$				
CB-ARMARoots	97.9%	5.55	94.9%	4.32
CB-ARMAPercentile	96.3%	5.25	91.6%	4.17
OB-ARMARoots	93.8%	4.51	87.9%	3.01
OB-ARMAPercentile	94.2%	4.84	87.2%	2.96
$n=100$				
CB-ARMARoots	96.6%	4.70	93.8%	3.90
CB-ARMAPercentile	96.3%	4.67	93.7%	3.93
OB-ARMARoots	93.3%	4.21	89.0%	3.05
OB-ARMAPercentile	93.4%	4.24	89.7%	3.09
$n=200$				
CB-ARMARoots	95.6%	4.66	91.3%	3.67
CB-ARMAPercentile	96.8%	4.88	92.0%	3.68
OB-ARMARoots	95.5%	4.83	90.8%	3.80
OB-ARMAPercentile	96.1%	4.80	91.8%	3.72
$n=400$				
CB-ARMARoots	95.1%	4.71	90.5%	3.69
CB-ARMAPercentile	93.7%	4.69	88.5%	3.72
OB-ARMARoots	94.5%	4.67	89.3%	3.66
OB-ARMAPercentile	93.6%	4.65	88.3%	3.69

Table 3: CVR and LEN for ARMA(1,2) with Normal innovations.

Normal	Nominal coverage 95%		Nominal coverage 90%	
	CVR	LEN	CVR	LEN
$n=25$				
CB-ARMARoots	97.2%	5.59	93.2%	4.55
CB-ARMAPercentile	92.7%	5.39	85.8%	4.42
OB-ARMARoots	93.0%	4.99	85.9%	4.15
OB-ARMAPercentile	86.4%	5.01	76.6%	4.18
$n=50$				
CB-ARMARoots	97.1%	5.31	95.1%	4.48
CB-ARMAPercentile	93.0%	5.51	88.4%	4.62
OB-ARMARoots	96.0%	4.94	93.5%	4.20
OB-ARMAPercentile	92.5%	4.95	87.5%	4.24
$n=75$				
CB-ARMARoots	97.6%	4.89	94.8%	4.04
CB-ARMAPercentile	93.8%	4.87	90.0%	4.12
OB-ARMARoots	94.4%	4.65	90.8%	3.91
OB-ARMAPercentile	92.0%	4.65	89.1%	3.91
$n=100$				
CB-ARMARoots	96.6%	4.51	92.1%	3.67
CB-ARMAPercentile	95.0%	4.42	90.1%	3.67
OB-ARMARoots	95.2%	4.00	90.0%	3.33
OB-ARMAPercentile	93.6%	4.00	88.2%	3.37
$n=200$				
CB-ARMARoots	94.7%	3.82	91.3%	3.41
CB-ARMAPercentile	94.5%	3.79	90.6%	3.31
OB-ARMARoots	92.0%	3.54	88.7%	3.20
OB-ARMAPercentile	92.2%	3.57	88.5%	3.19
$n=400$				
CB-ARMARoots	97.5%	4.46	93.8%	3.70
CB-ARMAPercentile	97.4%	4.46	93.7%	3.69
OB-ARMARoots	96.1%	4.51	94.1%	3.76
OB-ARMAPercentile	96.8%	4.49	93.8%	3.73

Table 4: CVR and LEN for ARMA(1,2) with Laplace innovations.

Laplace	Nominal coverage 95%		Nominal coverage 90%	
	CVR	LEN	CVR	LEN
$n=25$				
CB-ARMARoots	97.1%	5.17	93.0%	3.97
CB-ARMAPercentile	94.6%	5.97	89.9%	4.27
OB-ARMARoots	93.0%	4.06	88.2%	3.39
OB-ARMAPercentile	91.7%	4.06	84.5%	3.36
$n=50$				
CB-ARMARoots	96.1%	4.68	93.6%	4.01
CB-ARMAPercentile	96.0%	4.66	93.9%	4.03
OB-ARMARoots	94.9%	4.16	92.4%	3.58
OB-ARMAPercentile	94.6%	4.08	91.6%	3.47
$n=75$				
CB-ARMARoots	96.3%	4.90	92.6%	3.96
CB-ARMAPercentile	96.8%	4.84	93.5%	3.89
OB-ARMARoots	94.2%	4.80	90.2%	3.85
OB-ARMAPercentile	95.5%	4.77	92.3%	3.85
$n=100$				
CB-ARMARoots	97.8%	5.46	92.9%	3.87
CB-ARMAPercentile	96.6%	5.19	92.2%	3.87
OB-ARMARoots	96.6%	4.87	87.7%	3.28
OB-ARMAPercentile	93.7%	4.72	88.4%	3.32
$n=200$				
CB-ARMARoots	98.0%	5.62	94.4%	4.29
CB-ARMAPercentile	94.7%	5.80	89.5%	4.49
OB-ARMARoots	96.5%	4.67	91.5%	3.48
OB-ARMAPercentile	92.2%	4.62	84.6%	3.47
$n=400$				
CB-ARMARoots	97.6%	5.88	94.8%	4.67
CB-ARMAPercentile	98.0%	6.06	95.6%	4.85
OB-ARMARoots	96.6%	5.44	93.4%	4.34
OB-ARMAPercentile	96.1%	5.32	91.6%	4.26

Table 5: CVR and LEN for ARMA(2,2) with Normal innovations.

Normal	Nominal coverage 95%		Nominal coverage 90%	
	CVR	LEN	CVR	LEN
$n=25$				
CB-ARMARoots	96.0%	4.51	93.0%	3.76
CB-ARMAPercentile	94.9%	4.46	90.8%	3.65
OB-ARMARoots	93.8%	4.49	91.0%	3.70
OB-ARMAPercentile	93.3%	4.49	90.3%	3.73
$n=50$				
CB-ARMARoots	97.0%	4.34	90.5%	3.51
CB-ARMAPercentile	92.7%	3.94	88.4%	3.49
OB-ARMARoots	92.8%	3.77	89.2%	3.43
OB-ARMAPercentile	91.4%	3.76	87.1%	3.38
$n=75$				
CB-ARMARoots	98.0%	5.26	94.6%	4.29
CB-ARMAPercentile	95.9%	5.29	91.9%	4.39
OB-ARMARoots	96.5%	4.27	92.4%	3.58
OB-ARMAPercentile	95.5%	4.29	91.5%	3.60
$n=100$				
CB-ARMARoots	96.5%	5.51	93.5%	4.44
CB-ARMAPercentile	96.0%	5.96	92.4%	4.64
OB-ARMARoots	93.8%	4.31	89.5%	3.73
OB-ARMAPercentile	93.7%	4.25	89.1%	3.69
$n=200$				
CB-ARMARoots	96.5%	4.36	92.6%	3.67
CB-ARMAPercentile	97.1%	4.37	93.2%	3.66
OB-ARMARoots	96.7%	4.31	92.9%	3.63
OB-ARMAPercentile	96.5%	4.19	92.6%	3.55
$n=400$				
CB-ARMARoots	96.0%	4.07	91.3%	3.38
CB-ARMAPercentile	96.0%	4.08	91.2%	3.39
OB-ARMARoots	95.4%	4.05	91.0%	3.44
OB-ARMAPercentile	95.2%	4.03	89.9%	3.35

Table 6: CVR and LEN for ARMA(2,2) with Laplace innovations.

Laplace	Nominal coverage 95%		Nominal coverage 90%	
	CVR	LEN	CVR	LEN
$n = 25$				
CB-ARMARoots	94.5%	4.05	87.6%	3.00
CB-ARMAPercentile	94.0%	3.99	86.6%	2.88
OB-ARMARoots	93.7%	3.96	80.0%	2.27
OB-ARMAPercentile	93.8%	3.98	78.5%	2.28
$n = 50$				
CB-ARMARoots	98.8%	6.56	96.9%	4.97
CB-ARMAPercentile	98.9%	6.59	97.5%	5.33
OB-ARMARoots	98.0%	5.79	96.4%	4.68
OB-ARMAPercentile	98.0%	5.79	96.5%	4.69
$n = 75$				
CB-ARMARoots	91.9%	3.73	88.7%	3.15
CB-ARMAPercentile	92.3%	3.75	89.2%	3.18
OB-ARMARoots	91.6%	3.69	88.4%	3.13
OB-ARMAPercentile	91.8%	3.69	88.8%	3.14
$n = 100$				
CB-ARMARoots	95.6%	4.43	90.8%	3.42
CB-ARMAPercentile	95.5%	4.43	90.7%	3.39
OB-ARMARoots	94.9%	4.36	90.1%	3.35
OB-ARMAPercentile	94.7%	4.32	90.1%	3.36
$n = 200$				
CB-ARMARoots	95.5%	4.44	91.2%	3.44
CB-ARMAPercentile	95.3%	4.38	91.2%	3.45
OB-ARMARoots	94.9%	4.44	90.0%	3.31
OB-ARMAPercentile	95.3%	4.43	90.5%	3.32
$n = 400$				
CB-ARMARoots	94.8%	4.32	90.1%	3.42
CB-ARMAPercentile	94.1%	4.25	88.6%	3.31
OB-ARMARoots	91.6%	4.12	84.8%	3.25
OB-ARMAPercentile	93.6%	4.10	88.6%	3.24

Table 7: CVR and LEN for ARMA(3,1) with Normal innovations.

Normal	Nominal coverage 95%		Nominal coverage 90%	
	CVR	LEN	CVR	LEN
$n=25$				
CB-ARMARoots	97.2%	4.44	93.6%	3.76
CB-ARMAPercentile	95.4%	5.44	88.5%	4.19
OB-ARMARoots	96.2%	4.12	92.1%	3.54
OB-ARMAPercentile	94.7%	4.35	87.3%	3.56
$n=50$				
CB-ARMARoots	97.5%	4.38	92.7%	3.54
CB-ARMAPercentile	97.0%	4.29	91.2%	3.40
OB-ARMARoots	96.6%	4.23	89.6%	3.30
OB-ARMAPercentile	96.0%	4.16	89.6%	3.29
$n=75$				
CB-ARMARoots	95%	4.01	90.8%	3.39
CB-ARMAPercentile	96.2%	4.16	89.2%	3.27
OB-ARMARoots	94.2%	3.79	89.3%	3.22
OB-ARMAPercentile	94.0%	3.76	88.2%	3.15
$n=100$				
CB-ARMARoots	92.4%	3.57	86.0%	2.98
CB-ARMAPercentile	92.3%	3.56	86.2%	3.00
OB-ARMARoots	92.0%	3.45	86.0%	2.92
OB-ARMAPercentile	91.0%	3.38	86.4%	2.93
$n=200$				
CB-ARMARoots	90.9%	3.77	84.4%	3.07
CB-ARMAPercentile	90.8%	3.76	84.2%	3.06
OB-ARMARoots	89.8%	3.69	83.4%	3.00
OB-ARMAPercentile	89.8%	3.68	83.3%	2.99
$n=400$				
CB-ARMARoots	97.2%	4.39	90.1%	3.43
CB-ARMAPercentile	96.6%	4.37	88.0%	3.42
OB-ARMARoots	95.9%	4.11	88.4%	3.19
OB-ARMAPercentile	95.9%	4.15	87.2%	3.16

Table 8: CVR and LEN for ARMA(3,1) with Laplace innovations.

Laplace	Nominal coverage 95%		Nominal coverage 90%	
	CVR	LEN	CVR	LEN
$n = 25$				
CB-ARMARoots	90.9%	3.30	84.2%	2.76
CB-ARMAPercentile	84.7%	2.94	83.3%	2.72
OB-ARMARoots	84.2%	2.94	82.8%	2.72
OB-ARMAPercentile	84.2%	2.94	82.6%	2.72
$n = 50$				
CB-ARMARoots	94.0%	4.55	88.2%	3.65
CB-ARMAPercentile	92.4%	4.43	86.2%	3.61
OB-ARMARoots	92.7%	4.36	86.6%	3.48
OB-ARMAPercentile	91.7%	4.31	84.0%	3.40
$n = 75$				
CB-ARMARoots	97.0%	5.14	91.6%	3.44
CB-ARMAPercentile	96.9%	5.04	91.5%	3.43
OB-ARMARoots	96.9%	5.13	91.2%	3.41
OB-ARMAPercentile	97.0%	5.18	91.5%	3.44
$n = 100$				
CB-ARMARoots	97.7%	5.40	94.2%	4.11
CB-ARMAPercentile	97.7%	5.38	93.8%	3.99
OB-ARMARoots	97.5%	5.44	91.8%	3.95
OB-ARMAPercentile	97.1%	5.28	93.4%	3.93
$n = 200$				
CB-ARMARoots	95.0%	4.40	90.3%	3.36
CB-ARMAPercentile	93.8%	4.27	89.5%	3.38
OB-ARMARoots	94.5%	4.09	90.6%	3.43
OB-ARMAPercentile	94.2%	4.11	89.9%	3.36
$n = 400$				
CB-ARMARoots	96.2%	4.59	91.9%	3.47
CB-ARMAPercentile	96.1%	4.56	92.0%	3.49
OB-ARMARoots	96.4%	4.65	92.4%	3.50
OB-ARMAPercentile	96.4%	4.61	92.5%	3.48

5. CONCLUSIONS

In this paper we introduce the bootstrap algorithms for constructing the prediction intervals of ARMA (p, q) models with unknown orders based on the bootstrap distribution of orders. The asymptotic validity and asymptotic pertinence of the bootstrap prediction intervals are shown to hold true. We conduct simulations for several ARMA models using four bootstrap methods, i.e., CB-ARMARoots, CB-ARMAPercentile, OB-ARMARoots and OB-ARMAPercentile. From the simulation results we see that the proposed CB-ARMARoots algorithm outperforms the OB-ARMA methods using pre-estimated orders in terms of the coverage accuracy of the prediction intervals, especially when the true order of the ARMA model is low or when the sample size is small.

A. APPENDIX

Proof of Theorem 3.1: Note that if one can show that, as $n \to \infty$,

$$\sup_x \Big| P\big(X_{n+1} - \hat{X}_{n+1} \le x\big) - P\big(X^*_{n+1} - \hat{X}^*_{n+1} \le x\big) \Big| \ \to^p \ 0\,,$$

then standard results imply that the quantiles of the distribution of $X^*_{n+1} - \hat{X}^*_{n+1}$ can be used to consistently estimate the quantiles of the distribution of $X_{n+1} - \hat{X}_{n+1}$. This leads to the asymptotic validity of the prediction interval (2.4).

Let $Y_n = (X_n, ..., X_{n-p+1})$, $E_n = \big(\varepsilon_n, ..., \varepsilon_{n-q+1}\big)$. From Steps 6 and 7 of B-ARMARoots Algorithm, we obtain

$$\begin{aligned} X_{n+1} &= \big(Y_n, E_n\big)\, \theta + \varepsilon_{n+1}\,, \\ \hat{X}_{n+1} &= \big(Y_n, \hat{E}_n\big(\hat{\theta}^M\big)\big)\, \hat{\theta}^M\,, \\ X^*_{n+1} &= \big(Y_n, \hat{E}_n\big(\hat{\theta}^M\big)\big)\, \hat{\theta}^M + \varepsilon^*_{n+1}\,, \\ \hat{X}^*_{n+1} &= \big(Y_n, \hat{E}^*_n\big(\hat{\theta}^{M,*}\big)\big)\, \hat{\theta}^{M,*}\,. \end{aligned}$$

Thus

$$\begin{aligned} X_{n+1} - \hat{X}_{n+1} &= \big(Y_n, E_n\big)\, \theta - \big(Y_n, \hat{E}_n\big(\hat{\theta}^M\big)\big)\, \hat{\theta}^M + \varepsilon_{n+1} \\ &= \big(Y_n, E_n\big)\, \theta - \big(Y_n, E_n\big)\, \hat{\theta}^M + \big(Y_n, E_n\big)\, \hat{\theta}^M - \big(Y_n, \hat{E}_n\big(\hat{\theta}^M\big)\big)\, \hat{\theta}^M + \varepsilon_{n+1} \\ &:= I_1 + I_2 + \varepsilon_{n+1}\,. \end{aligned}$$

Since $\theta - \hat{\theta}^M = O_p\big(1/\sqrt{n}\big)$, $I_1 \to^p 0$ as $n \to \infty$.

By Lemma 2.1 of [9],

$$\varepsilon_j(\theta) - \hat{\varepsilon}_{j,n}\big(\hat{\theta}^M\big) \;=\; -Z'\big(j-1; \theta, \hat{\theta}^M\big)^{\mathsf{T}}\, \big(\theta - \hat{\theta}^M\big)$$

for any $1 \le j \le n$, where $Z'\big(j-1; \theta, \hat{\theta}^M\big)$ is uniformly bounded. Thus the $\sqrt{n}$-consistency of the M-estimator implies that

$$\varepsilon_j(\theta) - \hat{\varepsilon}_{j,n}\big(\hat{\theta}^M\big) \;=\; O_p\big(1/\sqrt{n}\big)\,. \tag{A.1}$$

From (2.9) of [9],

$$\varepsilon_j(\theta) = \varepsilon_j + (1+C)^{-j} O_p(1),$$

where C is a positive constant. For $n-q+1 \le j \le n$, $(1+C)^{-j} O_p(1) = O_p\big((1+C)^{-n}\big) = o_p\big(1/\sqrt{n}\big)$. Then

$$\varepsilon_j(\theta) = \varepsilon_j + o_p\big(1/\sqrt{n}\big). \tag{A.2}$$

(A.1) and (A.2) yield

$$\varepsilon_j - \hat{\varepsilon}_{j,n}\big(\hat{\theta}^M\big) = O_p\big(1/\sqrt{n}\big)$$

for $n-q+1 \le j \le n$. This implies that $E_n - \hat{E}_n\big(\hat{\theta}^M\big) = O_p\big(1/\sqrt{n}\big)$ and hence $I_2 \to^p 0$ as $n \to \infty$. That is

$$X_{n+1} - \hat{X}_{n+1} = \varepsilon_{n+1} + o_p(1).$$

Moreover, it follows from Theorems 3.1 and 4.1 of [9] that, as $n \to \infty$,

$$\sup_x \big|P\big(\varepsilon_{n+1} \le x\big) - P\big(\varepsilon^*_{n+1} \le x\big)\big| \to^p 0, \tag{A.3}$$

and

$$\hat{\theta}^{M,*} - \hat{\theta}^M \to^p 0.$$

Thus we obtain

$$X^*_{n+1} - \hat{X}^*_{n+1} = \varepsilon^*_{n+1} + o_p(1).$$

Now Slutsky's Lemma together with (A.3) concludes the proof Theorem 3.1. □

Proof of Theorem 3.2: By (A.3), Condition (i) in Definition 3.2 holds true. In view of the proof of Theorem 3.1, Condition (ii) also holds true. Moreover, Theorem 4.1 of [9] shows that $\sqrt{n}\,A_n$ and $\sqrt{n}\,A^*_n$ have the same asymptotic distribution. This implies Condition (iii). Finally, Condition (iv) follows from the causality of Model (2.1). This completes the proof of Theorem 3.2. □

Proof of Theorem 3.3: Let F_{n+1} be the distribution of X_{n+1} given $X_1, ..., X_n$, and F^*_{n+1} be the bootstrap distribution derived from CB-ARMARoots Algorithm. To prove the asymptotic validity, it suffices to show that

$$\sup_x \big|F^*_{n+1}(x) - F_{n+1}(x)\big| \to^p 0.$$

Note that

$$F_{n+1}(x) = \sum_{p_0,q_0} F_{n+1}\big(x \,|\, p_0, q_0\big)\, P\big(p=p_0,\, q=q_0\big),$$

$$F^*_{n+1}(x) = \sum_{p_0,q_0} F^*_{n+1}\big(x \,|\, p_0, q_0\big)\, P^*\big(p=p_0,\, q=q_0\big).$$

For any $\varepsilon > 0$, there exists (P, Q) such that

$$\sum_{p_0>P} \sum_{q_0} P\big(p=p_0,\, q=q_0\big) < \varepsilon, \qquad \sum_{p_0} \sum_{q_0>Q} P\big(p=p_0,\, q=q_0\big) < \varepsilon.$$

Obviously,

$$\sum_{p_0>P,\, q_0\le Q} P\big(p=p_0,\, q=q_0\big) < \varepsilon\,, \qquad \sum_{p_0\le P,\, q_0>Q} P\big(p=p_0,\, q=q_0\big) < \varepsilon$$

and

$$\sum_{p_0>P,\, q_0>Q} P\big(p=p_0,\, q=q_0\big) < \varepsilon\,.$$

Similarly, there exists such (P^*, Q^*) for $P^*\big(p=p_0,\, q=q_0\big)$. Let $(P_0, Q_0) = \big(\max(P, P^*), \max(Q, Q^*)\big)$. Thus

$$\begin{aligned}
&\sup_x \big|F_{n+1}(x) - F^*_{n+1}(x)\big| = \\
&\quad = \sup_x \left|\sum_{p_0,q_0}\Big\{F_{n+1}\big(x\,|\,p_0,q_0\big)\, P\big(p=p_0,\, q=q_0\big) - F^*_{n+1}\big(x\,|\,p_0,q_0\big)\, P^*\big(p=p_0,\, q=q_0\big)\Big\}\right| \le \\
&\le \sup_x \left|\sum_{p_0\le P_0,\, q_0\le Q_0}\Big\{F_{n+1}\big(x\,|\,p_0,q_0\big)\, P\big(p=p_0,\, q=q_0\big) - F^*_{n+1}\big(x\,|\,p_0,q_0\big)\, P^*\big(p=p_0,\, q=q_0\big)\Big\}\right| \\
&\quad + \sup_x \left|\sum_{p_0\le P_0,\, q_0> Q_0}\Big\{F_{n+1}\big(x\,|\,p_0,q_0\big)\, P\big(p=p_0,\, q=q_0\big) - F^*_{n+1}\big(x\,|\,p_0,q_0\big)\, P^*\big(p=p_0,\, q=q_0\big)\Big\}\right| \\
&\quad + \sup_x \left|\sum_{p_0> P_0,\, q_0\le Q_0}\Big\{F_{n+1}\big(x\,|\,p_0,q_0\big)\, P\big(p=p_0,\, q=q_0\big) - F^*_{n+1}\big(x\,|\,p_0,q_0\big)\, P^*\big(p=p_0,\, q=q_0\big)\Big\}\right| \\
&\quad + \sup_x \left|\sum_{p_0> P_0,\, q_0> Q_0}\Big\{F_{n+1}\big(x\,|\,p_0,q_0\big)\, P\big(p=p_0,\, q=q_0\big) - F^*_{n+1}\big(x\,|\,p_0,q_0\big)\, P^*\big(p=p_0,\, q=q_0\big)\Big\}\right|.
\end{aligned}$$

Observe that

$$\begin{aligned}
&\sup_x \left|\sum_{p_0\le P_0,\, q_0\le Q_0}\Big\{F_{n+1}\big(x\,|\,p_0,q_0\big)\, P\big(p=p_0,\, q=q_0\big) - F^*_{n+1}\big(x\,|\,p_0,q_0\big)\, P^*\big(p=p_0,\, q=q_0\big)\Big\}\right| \le \\
&\qquad \le \sup_x \left|\sum_{p_0\le P_0,\, q_0\le Q_0} F_{n+1}\big(x\,|\,p_0,q_0\big)\, \Big(P\big(p=p_0,\, q=q_0\big) - P^*\big(p=p_0,\, q=q_0\big)\Big)\right| \\
&\qquad\quad + \sup_x \left|\sum_{p_0\le P_0,\, q_0\le Q_0} \Big(F_{n+1}\big(x\,|\,p_0,q_0\big) - F^*_{n+1}\big(x\,|\,p_0,q_0\big)\Big)\, P^*\big(p=p_0,\, q=q_0\big)\right| \\
&\qquad \le \sum_{p_0\le P_0,\, q_0\le Q_0} \Big|P\big(p=p_0,\, q=q_0\big) - P^*\big(p=p_0,\, q=q_0\big)\Big| \\
&\qquad\quad + \sum_{p_0\le P_0,\, q_0\le Q_0} \sup_x \Big|F_{n+1}\big(x\,|\,p_0,q_0\big) - F^*_{n+1}\big(x\,|\,p_0,q_0\big)\Big|\,.
\end{aligned}$$

Hence we obtain

$$
\begin{aligned}
\sup_x \left|F_{n+1}(x) - F^*_{n+1}(x)\right| \;\le\; & \sum_{p_0\le P_0,\, q_0\le Q_0} \Big|P\big(p=p_0,\, q=q_0\big) - P^*\big(p=p_0,\, q=q_0\big)\Big| \\
& + \sum_{p_0\le P_0,\, q_0\le Q_0} \sup_x \Big|F_{n+1}\big(x\,|\,p_0,q_0\big) - F^*_{n+1}\big(x\,|\,p_0,q_0\big)\Big| \\
& + \sum_{p_0\le P_0,\, q_0> Q_0} P\big(p=p_0,\, q=q_0\big) + \sum_{p_0> P_0,\, q_0\le Q_0} P\big(p=p_0,\, q=q_0\big) \\
& + \sum_{p_0\le P_0,\, q_0> Q_0} P^*\big(p=p_0,\, q=q_0\big) + \sum_{p_0> P_0,\, q_0\le Q_0} P^*\big(p=p_0,\, q=q_0\big) \\
& + \sum_{p_0> P_0,\, q_0> Q_0} P\big(p=p_0,\, q=q_0\big) + \sum_{p_0> P_0,\, q_0> Q_0} P^*\big(p=p_0,\, q=q_0\big) \\
\le\; & \sum_{p_0\le P_0,\, q_0\le Q_0} \Big|P\big(p=p_0,\, q=q_0\big) - P^*\big(p=p_0,\, q=q_0\big)\Big| \\
& + \sum_{p_0\le P_0,\, q_0\le Q_0} \sup_x \Big|F_{n+1}\big(x\,|\,p_0,q_0\big) - F^*_{n+1}\big(x\,|\,p_0,q_0\big)\Big| + 6\,\varepsilon\,.
\end{aligned}
\tag{A.4}
$$

[5] showed that

$$\Big|P\big(p=p_0,\, q=q_0\big) - P^*\big(p=p_0,\, q=q_0\big)\Big| \;\to^p\; 0\,,$$

and Theorem 3.1 implies that

$$\sup_x \Big|F_{n+1}\big(x\,|\,p_0,q_0\big) - F^*_{n+1}\big(x\,|\,p_0,q_0\big)\Big| \;\to^p\; 0\,.$$

These, together with (A.4) and the arbitrariness of ε, yield that

$$\sup_x \left|F_{n+1}(x) - F^*_{n+1}(x)\right| \;\to^p\; 0\,.$$

Along similar lines of the proof of Theorem 3.2, the asymptotic pertinence of the prediction interval (2.5) also holds true. □

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