REVSTAT – Statistical Journal
Volume 2, Number 2, November 2004

EXTENSIONS OF KATZ–PANJER FAMILIES OF DISCRETE DISTRIBUTIONS *

Authors: Dinis D. Pestana
– Departamento de Estatística e Investigação Operacional,
Faculdade de Ciências da Universidade de Lisboa, Bloco C6, Piso 4,
Campo Grande, 1749-016 Lisboa, Portugal, e
CEAUL – Centro de Estatística e Aplicações da Universidade de Lisboa
(dinis.pestana@fc.ul.pt)

Sílvio F. Velosa
– Departamento de Matemática e Engenharias, Universidade da Madeira,
Campus Universitário da Penteada, 9000-390 Funchal, Portugal, e
CEAUL – Centro de Estatística e Aplicações da Universidade de Lisboa
(sfilipe@uma.pt)

Received: July 2004 Revised: September 2004 Accepted: September 2004

Abstract:

- Let $N_{\alpha,\beta,\gamma}$ be a discrete random variable whose probability atoms $\{p_n\}_{n\in\mathbb{N}}$ satisfy $\frac{f(n+1)}{f(n)} = \alpha + \beta\,\frac{\mathbb{E}(U_0^n)}{\mathbb{E}(U_\gamma^n)}$, $n=0,1,...,$ for some $\alpha,\beta\in\mathbb{R}$, where $U_\gamma \frown \mathit{Uniform}(\gamma,1)$, $\gamma\in(-1,1]$. When $\gamma\to 1$, $U_\gamma\to U_1$, the degenerate random variable with unit mass at 1, and the above iterative expression is $\frac{p_{n+1}}{p_n} = \alpha + \frac{\beta}{n+1}$ for $n=k,k+1,...,$ used by Katz and by Panjer ($k=0$), by Sundt and Jewell and by Willmot ($k=1$) and, for general $k\in\mathbb{N}$, by Hess, Lewald and Schmidt.
We investigate the case $U_\gamma \frown \mathit{Uniform}(\gamma,1)$ with $\gamma\in(-1,1)$ in detail for $\alpha=0$. We then construct classes $\mathcal{C}_\gamma$ of discrete infinitely divisible randomly stopped sums such that $N_{0,\beta,\gamma}\in\mathcal{C}_\gamma$. $\mathcal{C}_0$ is the class of compound geometric random variables, $\mathcal{C}_1$ is the class of compound Poissons, and $|\gamma_1|<\gamma_2\le 1$ implies $\mathcal{C}_{\gamma_1}\subset\mathcal{C}_{\gamma_2}\subseteq\mathcal{C}_1$.

Key-Words:

- *Poisson stopped sums (compound Poisson); geometric stopped sums (compound geometric); Panjer's algorithm.*

AMS Subject Classification:

- 60G50, 60E10, 91B30.

*Research partially suported by FCT/POCTI/FEDER.

1. INTRODUCTION

Let us consider the discrete random variables $N_{\alpha,\beta}$ whose probability mass functions (p.m.f.) $\{f_{N_{\alpha,\beta}}(n)\}_{n\in\mathbb{N}}$ satisfy

$$f_{N_{\alpha,\beta}}(n+1) = \left(\alpha + \frac{\beta}{n+1}\right) f_{N_{\alpha,\beta}}(n) , \qquad \alpha,\beta \in \mathbb{R}, \quad n = 0,1,... \tag{1.1}$$

From (1.1) it follows that $f_{N_{\alpha,\beta}}(n) = f_{N_{\alpha,\beta}}(0) \prod_{k=1}^{n} \left(\alpha + \frac{\beta}{k}\right)$. In particular,

$$f_{N_{\alpha,0}}(n) = f_{N_{\alpha,0}}(0)\, \alpha^n = (1-\alpha)\, \alpha^n \implies N_{\alpha,0} \frown \textit{Geometric}(1-\alpha) ,$$

and we may write

$$f_{N_{\alpha,0}}(n+1) \;=\; \alpha\, f_{N_{\alpha,0}}(n) \;=\; \sum_{k=0}^{n} f_{N_{\alpha,0}}(n)\, r_{n-j} , \tag{1.2}$$

where $r_0 = \alpha$ is the ratio of a geometric series and $r_1 = \cdots = r_n = 0$.

On the other hand,

$$f_{N_{0,\beta}}(n) = f_{N_{0,\beta}}(0) \prod_{k=1}^{n} \frac{\beta}{n} = f_{N_{0,\beta}}(0)\, \frac{\beta^n}{n!} \;=\; \mathrm{e}^{-\beta}\, \frac{\beta^n}{n!} \implies N_{\alpha,0} \frown \textit{Poisson}(\beta) ,$$

and we may write

$$(n+1)\, f_{N_{0,\beta}}(n+1) \;=\; \beta\, f_{N_{0,\beta}}(n) \;=\; \sum_{k=0}^{n} f_{N_{0,\beta}}(n)\, r_{n-j} , \tag{1.3}$$

where $r_0 = \beta$ and $r_1 = \cdots = r_n = 0$. Note that similar expressions do not hold for randomly stopped sums $S_{N_{\alpha,\beta}} = S_{N_{\alpha,\beta}}(Y) = \sum_{k=1}^{N_{\alpha,\beta}} Y_k$, where the summands Y_k are i.i.d. and independent of the subordinator $N_{\alpha,\beta}$, with p.m.f. satisfying (1.1), whenever both $\alpha \neq 0$ and $\beta \neq 0$. However, for geometric stopped sums $\sum_{k=1}^{N_{\alpha,0}} Y_k$ and for Poisson stopped sums, $\sum_{k=1}^{N_{0,\beta}} Y_k$ (i.e., when either $\beta = 0$ or $\alpha = 0$) we get nice similar expressions, with the $r_k \geq 0$ and convergence of $\sum_{k=0}^{\infty} r_k$, in the case of geometric stopped sums, and convergence of $\sum_{k=0}^{\infty} \frac{r_k}{k+1}$, for Poisson stopped sums. In the definition of randomly stopped sums, $\mathbb{P}\big[S_{N_{\alpha,\beta}}{=}0 \mid N_{\alpha,\beta}{=}0\big] = 1$, and therefore $\mathbb{P}[S_{N_{\alpha,\beta}}{=}0] = \mathbb{P}[N_{\alpha,\beta}{=}0] = f_{N_{\alpha,\beta}}(0)$ whenever $\mathbb{P}[Y_k > 0] = 1$).

Panjer (1981) has remarked that the discrete (nondegenerate) random variables whose p.m.f.'s satisfy equation (1.1) are

- $N_{0,\beta} \frown Poisson(\beta),\ \beta > 0,$
- $N_{\alpha,\beta} \frown Binomial\Big(-1-\frac{\beta}{\alpha},\ \frac{\alpha}{\alpha-1}\Big)$, in case $\alpha < 0$ and $-\frac{\beta}{\alpha} \in \mathbb{N}^+$, and
- $N_{\alpha,\beta} \frown NegativeBinomial\Big(\frac{\alpha+\beta}{\alpha},\ 1-\alpha\Big)$ if $\alpha \in (0,\,1)$ and $\alpha+\beta > 0$.

The dispersion index $\frac{\mathrm{var}(N_{\alpha,\beta})}{\mathbb{E}(N_{\alpha,\beta})} = \frac{1}{1-\alpha}$ is less than 1 (underdispersion) for the binomial and greater than 1 (overdispersion) for the negative binomial. On the other hand, $N_{0,\beta} \frown Poisson(\beta)$ is a yardstick, with dispersion index 1. We denote $\mathbf{\Pi}$ the class of random variables $N_{\alpha,\beta}$ described above.

These random variables play an important role as subordinators in randomly stopped sums. Compound or generalized random variables (other names traditionally given to $S_{N_{\alpha,\beta}}$, cf. the discussion on terminology in Johnson, Kotz and Kemp, 1992) are at the core of branching processes and many other subjects where the aim is to obtain the distribution of randomly stopped sums, namely in the study of aggregate claims in the risk process, see Klugman, Panjer and Willmot (1998) and Rólski, Schmidli, Schmidt and Teugels (1999).

Katz (1965) had used an iterative expression equivalent to (1.1) to organize a coordinated presentation of count distributions. Panjer's (1981) pathbreaking result has been to use the iterative expression satisfied by the p.m.f. of the subordinator $N_{\alpha,\beta}$ to get an iterative algorithm to compute the density function (probability mass function or probability density function) of $S_{N_{\alpha,\beta}}$. This is used in section 2 to establish characterization theorems for infinitely divisible and for geometric infinitely divisible generating functions.

In section 3, we investigate discrete random variables $N_{\alpha,\beta,\gamma}$ whose probability mass function (p.m.f.) $\{p_n\}_{n\in\mathbb{N}}$ satisfies the more general relation

$$(1.4) \qquad \frac{f_{N_{\alpha,\beta,\gamma}}(n+1)}{f_{N_{\alpha,\beta,\gamma}}(n)} = \alpha + \beta\,\frac{\mathbb{E}(U_0^n)}{\mathbb{E}(U_\gamma^n)} = \alpha + \frac{\beta}{\sum\limits_{k=0}^{n} \gamma^k}\,, \qquad \alpha,\beta \in \mathbb{R}, \quad n = 0,1,...$$

where $U_\gamma \frown Uniform(\gamma,\,1)$, $\gamma \in (-1,\,1)$. As

$$(1.5) \qquad \mathbb{E}(U_\gamma^n) = \frac{1}{n+1}\,\frac{1-\gamma^{n+1}}{1-\gamma} \underset{\gamma\to 1}{\longrightarrow} 1\,,$$

Panjer's class corresponds to the degenerate limit case, letting $\gamma \to 1$ so that $U_\gamma \to U_1$, the degenerate random variable with unit mass at 1.

When $\alpha = 0$, the iterative expression for the p.m.f. of $N_{0,\beta,\gamma}$ verifies

$$\frac{1-\gamma^{n+1}}{1-\gamma} f_{N_{\alpha,\beta,\gamma}}(n+1) = \sum_{k=0}^{n} f_{N_{\alpha,\beta,\gamma}}(k)\, r_{n-k} \tag{1.6}$$

with $r_0 = \beta$ and $r_1 = \cdots = r_n = 0$, of which (1.2) and (1.3) aren't but the cases $\gamma = 0$ and $\gamma = 1$, respectively. We shall investigate the classes $\mathcal{C}_\gamma$ of randomly stopped sums $\sum_{k=0}^{N_{0,\beta,\gamma}} Y_k$, whose members satisfy (1.6) for nonnegative r_k, with of $\sum_{k=0}^{\infty} r_k < \infty$.

In section 4 we show that when $|\gamma_1| < \gamma_2 \le 1$, $\mathcal{C}_{\gamma_1} \subset \mathcal{C}_{\gamma_2}$. Also, for $\gamma \in [0,1]$, the classes $\mathcal{C}_\gamma$ form an increasing chain of classes of infinitely divisible random variables, spanning from $\mathcal{C}_0$, the class of discrete geometric stopped sums, to $\mathcal{C}_1$, the class of discrete Poisson stopped sums.

Many of these results rely on properties of absolutely monotone functions scattered in the literature, that we shall discuss in section 2 below in conjunction with Panjer theory. Ospina and Gerber (1987) remarked that the representation theorem for the generating functions of discrete stopped Poisson sums (discrete infinitely divisible laws) follows from Panjer's theory, and the same is true for the representation of geometric infinitely divisible generating functions, see section 2, and for wider classes of generating functions whose bearing on general p-infinite divisibility is worth noting. This will be further discussed in the concluding section.

2. BASIC RESULTS

Let $\mathcal{G}(s) = \sum_{n=0}^{\infty} f(n) s^n$, $s \in [0,r)$, be the generating function of the sequence $\{f(n)\}_{n\in\mathbb{N}}$; in other words, $f(n) = \frac{\mathcal{G}^{(n)}(0)}{n!}$, $n \in \mathbb{N}$.

If $p_n \ge 0$, $n \in \mathbb{N}$, then $\mathcal{G}^{(n)}(s) \ge 0$, $s \in [0,r)$, and we say that $\mathcal{G}$ is absolutely monotone (abs. mon.) in $[0,r)$. If there exists $r > 0$ such that $\mathcal{G}$ is abs. mon. in $[0,r)$, we say that the function $\mathcal{G}$ is abs. mon. (Bernstein, 1928).

We refer to Widder (1946, chapt. IV) and to Feller (1968, chap. XI) for basic information on absolutely monotone functions and generating functions; Skellam and Shelton (1957) or Srivastava and Manocha (1984) provide a thorough discussion. It is obvious that the sum or the product of abs. mon. functions is abs. mon.; we shall need the following results:

1. $\mathcal{G}$ is abs. mon. $\iff \mathcal{G}(0) \ge 0$ and $\frac{d\mathcal{G}}{ds}$ is abs. mon. $\iff \frac{d}{ds}[s\,\mathcal{G}(s)]$ is abs. mon. (since $p_n \ge 0$ iff $(1+n)\, p_n \ge 0$).

2. Let $\gamma \in (-1, 1)$; then, $\mathcal{G}$ abs. mon. $\Longleftrightarrow$ $\mathcal{G}(s) - \gamma\,\mathcal{G}(\gamma s)$ abs. mon. (it is sufficient to note that $p_n \geq 0 \Longleftrightarrow p_n(1-\gamma^{n+1}) \geq 0$).

Let $|\gamma| \leq \eta < 1$; then, $\mathcal{G}$ abs. mon. $\Longrightarrow$ $\eta\,\mathcal{G}(\eta s) - \gamma\,\mathcal{G}(\gamma s)$ abs. mon. ($p_n \geq 0$ implies $p_n(\eta^{n+1} - \gamma^{n+1}) \geq 0$).

Note that $\eta\,\mathcal{G}(\eta s) - \gamma\,\mathcal{G}(\gamma s)$ is no longer abs. mon. if $-1 < \eta < \gamma \leq 0$.

3. If $\mathcal{G}_1$ is abs. mon. in $[0, r_1)$, $\mathcal{G}_2$ is abs. mon. in $[0, r_2)$, and $\mathcal{G}_2(s) < r_1$ for all $s \in [0, r_2)$, the compound function $\mathcal{G}_1 \circ \mathcal{G}_2 = \mathcal{G}_1(\mathcal{G}_2)$ is abs. mon. in $[0, r_2)$. In particular:

(**a**) As $\mathcal{G}_1(s) = \mathrm{e}^s$ is the generating function of $p_n = \frac{1}{n!}$, $\mathcal{G}_2$ abs. mon. implies that $(\mathcal{G}_1 \circ \mathcal{G}_2)(s) = \mathrm{e}^{\mathcal{G}_2(s)}$ is abs. mon.

(**b**) As $\mathcal{G}_1(s) = \frac{1}{1-s}$ is the generating function of $p_n = 1$, $\mathcal{G}_2$ abs. mon. in $[0, r_2)$ with $\mathcal{G}_2(s) < 1$ for $s \in [0, r_2)$ implies that $(\mathcal{G}_1 \circ \mathcal{G}_2)(s) = \frac{1}{1-\mathcal{G}_2(s)}$ is abs. mon.

Let us now consider the randomly stopped sum $S_{N_{\alpha,\beta}} = \sum_{k=1}^{N_{\alpha,\beta}} Y_k$, where $Y_k \stackrel{\mathrm{d}}{=} Y$, $k=1,2,...$, are i.i.d. counting random variables, with p.m.f. $\{f_Y(n)\}_{n\in\mathbb{N}}$, independent of the Panjer subordinator $N_{\alpha,\beta}$.

As

$$\mathbb{E}\left[\frac{k}{n+1}\,Y_1 \,\middle|\, \sum_{i=1}^{k} Y_i = n+1\right] = 1$$

and

$$\mathbb{P}\left[Y_1 = j \,\middle|\, \sum_{i=1}^{k} Y_i = n+1\right] = \frac{f_Y(j)\, f_Y^{\star(k-1)}(n+1-j)}{f_Y^{\star k}(n+1)}\,, \qquad j = 0, ..., n+1$$

(Rólski *et al.*, 1999, p. 119), where as usual $f^{\star k}$ denotes the k-fold convolution ($f^{1*} = f$, $f^{\star k} = f * f^{\star(k-1)}$), it follows that the probability mass function of a Poisson stopped sums ($N_{0,\beta}$, $\beta > 0$) verifies

$$\begin{aligned}(n+1)\, f_{S_{N_{0,\alpha}}}(n+1) &= \sum_{k=0}^{n} f_{S_{N_{0,\alpha}}}(k)\,\beta\,(n+1-k)\, f_Y(n+1-k)\\ &= \sum_{k=0}^{n} f_{S_{N_{0,\alpha}}}(k)\, r_{n-k}\,,\end{aligned} \tag{2.1}$$

with $r_k = \beta\,(k+1)\, f_Y(k+1) \geq 0$, $k = 0, 1, ...$, and it therefore follows that the generating function $\mathcal{H}_{N_{0,\beta}}(s) = \sum_{k=0}^{\infty} r_k\, s^k$ of the $\{r_k\}_{k\in\mathbb{N}}$ is absolutely monotone, with $\sum_{k=0}^{\infty} \frac{r_k}{k+1} = \sum_{k=0}^{\infty} \beta\, f_Y(k+1) = \beta\,(1 - f_Y(0))$. Assuming that $f_Y(0) = 0$ (i.e., enforcing a unique representation by fixing this free parameter), multiplying both sides of (1.6) by s^n and summing for $n = 0, 1, ...,$ we get,

$$\mathcal{G}_{S_{N_{0,\beta}}}(s) = \exp\left[\beta\left(\frac{1}{\beta}\sum_{k=0}^{\infty}\frac{r_k}{k+1}s^{k+1} - 1\right)\right] = \mathrm{e}^{\beta[\mathcal{P}(s)-1]} , \tag{2.2}$$

where $\mathcal{P}(s) = \frac{1}{\beta}\sum_{k=0}^{\infty}\frac{r_k}{k+1}s^{k+1}$ is a (unique) p.g.f., such that $\mathcal{P}(0) = 0$.

On the other hand, for geometric stopped sums ($N_{\alpha,0}$, $0<\alpha<1$) we get

$$f_{S_{N_{\alpha,0}}}(n+1) = \sum_{k=0}^{n} f_{S_{N_{\alpha,0}}}(k)\, r_{n-k} \tag{2.3}$$

where $r_k = \frac{\alpha f_Y(k+1)}{1-\alpha f_Y(0)}$. As in the treatment of Poisson stopped sums, we may get a unique representation theorem by letting the free parameter $f_Y(0)=0$, which implies $\sum_{k=0}^{\infty} r_k = \alpha$, multiplying both sides of (2.3) by s^n and summing for $n = 0, 1, ...$. In terms of generating functions,

$$\frac{\mathcal{G}_{S_N}(s) - f_{N_{\alpha,0}}(0)}{s} = \mathcal{G}_{S_N}(s)\,\mathcal{H}_{N_{\alpha,0}}(s)$$

where $\mathcal{H}_{N_{\alpha,0}}$ is the generating function of $\{r_n\}_{n\in\mathbb{N}}$, which are all nonnegative, with $\sum_{n=0}^{\infty} r_n = \alpha \in (0,1)$, i.e. $\mathcal{H}_{N_{\alpha,0}}$ is abs. mon. From $\mathcal{G}_{S_N}(1) = \frac{p_0}{1-\mathcal{H}_{N_{\alpha,0}}(1)} = \frac{p_0}{1-\alpha} = 1$, it follows that

$$\mathcal{G}_{S_N}(s) = \frac{p_0}{1 - s\,\mathcal{H}_{N_{\alpha,0}}(s)} = \frac{1-\alpha}{1 - s\sum_{k=0}^{\infty} r_k\, s^k} = \frac{1-\alpha}{1-\alpha\,\mathcal{P}(s)}$$

where $\mathcal{P}(s) = \sum_{k=0}^{\infty}\frac{r_k}{\alpha}s^{k+1}$, such that $\mathcal{P}(0) = 0$, is a p.g.f., because it is abs. mon. and $\mathcal{P}(1) = 1$. In other words, Panjer's iteration also provides a straightforward proof of the representation theorem for geometric infinitely divisible lattice distributions.

We record these representation theorems for the sake of the corollaries that we then establish, which will be instrumental in the proof of the extensions in sections 3 and 4.

Theorem 2.1. *The p.g.f.* $\mathcal{G}_{S_{N_{0,\beta}}}$ *of a discrete Poisson stopped sum such that* $\mathbb{P}\big[S_{N_{0,\beta}}=0\big] = f_{S_{N_{0,\beta}}}(0) > 0$ *has a unique representation* $\mathcal{G}_{S_{N_{0,\beta}}}(s) = \mathrm{e}^{\beta[\mathcal{P}(s)-1]}$, *where* $\mathcal{P}$ *is a p.g.f. such that* $\mathcal{P}(0) = 0$, *and* $\beta = -\ln \mathcal{G}_{S_{N_{0,\beta}}}(0)$.

The p.g.f. $\mathcal{G}_{S_{N_{\alpha,0}}}$ *of a discrete geometric stopped sum such that* $\mathbb{P}[S_N=0] = f_{S_{N_{\alpha,0}}}(0) > 0$ *has a unique representation* $\mathcal{G}_{S_{N_{\alpha,0}}}(s) = \frac{1-\alpha}{1-\alpha\,\mathcal{P}(s)}$, *where* $\mathcal{P}$ *is a p.g.f. such that* $\mathcal{P}(0) = 0$, *and* $\alpha = 1 - \mathcal{G}_{S_{N_{\alpha,0}}}(0)$.

Observe also that $\exp\Big(1-\frac{1}{\mathcal{G}_{S_{N_{\alpha,0}}}}\Big) = \mathrm{e}^{\frac{\alpha}{1-\alpha}[\mathcal{P}(s)-1]} = \mathcal{G}_{S_{N_{0,\frac{\alpha}{1-\alpha}}}}(s)$. On the other hand, $\dfrac{1}{1-\ln\Big(\mathcal{G}_{S_{N_{0,\beta}}}(s)\Big)} = \dfrac{1-\frac{\beta}{\beta+1}}{1-\frac{\beta}{\beta+1}\,\mathcal{P}(s)} = \mathcal{G}_{S_{N_{\frac{\beta}{\beta+1},0}}}(s)$.

Corolary 2.1.1.

(**1**) Let $\mathcal{G}$ be a probability generating function such that $\mathcal{G}(0)>0$; then, $\mathcal{G}$ is the p.g.f. of a discrete Poisson stopped sum iff $\frac{\mathcal{G}'(s)}{\mathcal{G}(s)}$ is abs. mon.

(**2**) Let $\mathcal{G}$ be a p.g.f. such that $\mathcal{G}(0)>0$, and $\gamma\in(-1,1)$. If $\mathcal{G}$ is the p.g.f. of a discrete Poisson stopped sum, then $\frac{\mathcal{G}(s)}{\mathcal{G}(\gamma s)}$ is abs. mon., and $\mathcal{G}_\gamma(s) = \frac{\mathcal{G}(\gamma)\,\mathcal{G}(s)}{\mathcal{G}(\gamma s)}$ is also the p.g.f. of a Poisson stopped sum.

(**3**) Let $\mathcal{G}$ be a p.g.f. such that $\mathcal{G}(0)>0$, and $|\gamma_1|\le\gamma_2<1$. If $\mathcal{G}$ is the p.g.f. of a discrete Poisson stopped sum, then $\frac{\mathcal{G}(\gamma_2 s)}{\mathcal{G}(\gamma_1 s)}$ is abs. mon. and $\mathcal{G}_{\gamma_1,\gamma_2}(s) = \frac{\mathcal{G}(\gamma_1)}{\mathcal{G}(\gamma_2)}\,\frac{\mathcal{G}(\gamma_2 s)}{\mathcal{G}(\gamma_1 s)}$ is also the p.g.f. of a Poisson stopped sum.

(**4**) Any discrete geometric stopped sum such that $\mathbb{P}[S_N=0]=\tilde{p}_0>0$ is a Poisson stopped sum, i.e. infinitely divisible.

Proof: (**1**) From Theorem 2.1 we know that $\mathcal{G}$, with $\mathcal{G}(0)>0$, is the p.g.f. of a Poisson stopped sum iff $\frac{\mathcal{G}'(s)}{\mathcal{G}(s)} = \mathcal{H}_{N_{0,\beta}}(s) = \sum\limits_{k=0}^{\infty} r_k s^k$, where $r_k = \beta\,(k+1)\,f_Y(k+1)\ge 0$, $k=0,1,\ldots$, and therefore its generating function $\mathcal{H}_{N_{0,\beta}}(s) = \sum\limits_{k=0}^{\infty} r_k\, s^k$ is absolutely monotone.

(**2**) From formula (2.2), we see that $\mathcal{G}(s)>0$ for all s, therefore $\frac{\mathcal{G}(s)}{\mathcal{G}(\gamma s)}\ge 1$ if $0\le s\le 1$. On the other hand, $\frac{\mathrm{d}}{\mathrm{d}s}\Big[\ln\frac{\mathcal{G}(s)}{\mathcal{G}(\gamma s)}\Big] = \frac{\mathcal{G}'(s)}{\mathcal{G}(s)} - \gamma\,\frac{\mathcal{G}'(\gamma s)}{\mathcal{G}(\gamma s)}$ is abs. mon., by 2.1.1.(1) and property 2 of abs. mon. functions. As $\ln\frac{\mathcal{G}(s)}{\mathcal{G}(\gamma s)}$ is nonnegative for $s=0$, it is also abs. mon., by property 1 of abs. mon. functions. From property 3(a) of abs. mon. functions, it follows that $\frac{\mathcal{G}(s)}{\mathcal{G}(\gamma s)}$ is abs. mon. Since $\mathcal{G}_\gamma(0)=\mathcal{G}(\gamma)>0$, $\mathcal{G}_\gamma(1)=1$, and $\frac{\mathcal{G}'_\gamma(s)}{\mathcal{G}_\gamma(s)} = \frac{\mathrm{d}}{\mathrm{d}s}\Big[\ln\frac{\mathcal{G}(s)}{\mathcal{G}(\gamma s)}\Big]$ is abs. mon., we conclude that $\mathcal{G}_\gamma$ is the p.g.f. of a Poisson stopped sum.

(**3**) By 2.1.1.(2), $\frac{\mathcal{G}(\gamma_2 s)}{\mathcal{G}(\gamma_1 s)}$ is abs. mon., and by property 2 of abs. mon. functions $\frac{\mathcal{G}'_{\gamma_1,\gamma_2}(s)}{\mathcal{G}_{\gamma_1,\gamma_2}(s)} = \gamma_2\,\frac{\mathcal{G}'(\gamma_2 s)}{\mathcal{G}(\gamma_2 s)} - \gamma_1\,\frac{\mathcal{G}'(\gamma_1 s)}{\mathcal{G}(\gamma_1 s)}$ is abs. mon. Since $\mathcal{G}_{\gamma_1,\gamma_2}(0) = \frac{\mathcal{G}(\gamma_1)}{\mathcal{G}(\gamma_2)}>0$ and $\mathcal{G}_{\gamma_1,\gamma_2}(1)=1$, it follows that $\mathcal{G}_{\gamma_1,\gamma_2}(s) = \frac{\mathcal{G}(\gamma_1)}{\mathcal{G}(\gamma_2)}\,\frac{\mathcal{G}(\gamma_2 s)}{\mathcal{G}(\gamma_1 s)}$ is the p.g.f. of a Poisson stopped sum.

(**4**) As we have seen, $\mathcal{G}$ with $\mathcal{G}(0) > 0$ is the p.g.f. of a discrete geometric stopped sum iff $\mathcal{G}(s) = \frac{\mathcal{G}(0)}{1 - s\,\mathcal{H}_{N_{\alpha,0}}(s)}$, where $\mathcal{H}_{N_{\alpha,0}}(s) < 1$ for $s \in [0, 1)$ is abs. mon. As

$$\frac{\mathcal{G}'(s)}{\mathcal{G}(s)} = \frac{\dfrac{\mathcal{G}(0)\,\frac{\mathrm{d}}{\mathrm{d}s}\big[s\,\mathcal{H}_{N_{\alpha,0}}(s)\big]}{\big(1 - s\,\mathcal{H}_{N_{\alpha,0}}(s)\big)^2}}{\dfrac{\mathcal{G}(0)}{1 - s\,\mathcal{H}_{N_{\alpha,0}}(s)}} = \frac{\frac{\mathrm{d}}{\mathrm{d}s}\big[s\,\mathcal{H}_{N_{\alpha,0}}(s)\big]}{1 - s\,\mathcal{H}_{N_{\alpha,0}}(s)}\ ,$$

and from property 1 of abs. mon. functions $\frac{\mathrm{d}}{\mathrm{d}s}\big[s\,\mathcal{H}_{N_{\alpha,0}}(s)\big]$ is abs. mon., from property 3(b) we know that $\frac{1}{1 - s\,\mathcal{H}_{N_{\alpha,0}}(s)}$ is abs. mon., and the product of abs. mon. functions is abs. mon., it follows that $\frac{\mathcal{G}'(s)}{\mathcal{G}(s)}$ is abs. mon. From part (1) of Corollary 2.1.1., it follows that $\mathcal{G}$ is the p.g.f. of a Poisson stopped sum. □

If the probability generating function $\mathcal{G}_Y$ of $Y \frown F_Y$ depends on the parameter θ so that $\mathcal{G}_Y(s|\,k\theta) = [\mathcal{G}_Y(s|\,\theta)]^k$, then $F_X \vee F_Y = F_Y \underset{K}{\wedge} F_X$, where $\vee$ denotes the stopped sum of Y independent copies of X, and $\underset{K}{\wedge}$ denotes the mixture of $Y|\,K$, with mixing distribution F_X (Gurland, 1957). Therefore the class of discrete Poisson stopped sums coincides with the class of discrete mixtures of Poisson random variables. In what mixtures of geometric random variables and geometric stopped sums, the former is stricly included in the later.

3. EXTENSIONS

We now investigate the nondegenerate discrete random variables $N_{\alpha,\beta,\gamma}$ whose probability mass function $\{p_n\}_{n\in\mathbb{N}}$ satisfies

$$\text{(3.1)}\qquad \frac{p_{n+1}}{p_n} = \alpha + \beta\,\frac{\mathbb{E}(U_0^n)}{\mathbb{E}(U_\gamma^n)} = \alpha + \beta\,\frac{1-\gamma}{1-\gamma^{n+1}} \qquad \text{for}\ \ n = 0, 1, ..., \quad \alpha, \beta \in \mathbb{R}\ ,$$

where $U_\gamma \frown \mathit{Uniform}(\gamma, 1)$, $\gamma \in (-1, 1)$, with $p_0 > 0$. If $\gamma = 0$, all possible solutions are geometric random variables, and when $\gamma \to 1$ we get Panjer's class of counting distributions.

For a nondegenerate solution of (3.1) with infinite support to exist, we must have

$$\alpha + \beta\,\frac{\mathbb{E}(U_0^n)}{\mathbb{E}(U_\gamma^n)} = \alpha + \beta\,\frac{1-\gamma}{1-\gamma^{n+1}} > 0$$

for every integer n. According to the signs of β and γ, the infimum of this factor is either $\alpha + \beta$ (for $n = 0$), $\alpha + \frac{\beta}{1+\gamma}$ (for $n = 1$), or $\alpha + \beta(1-\gamma)$ (when $n \to \infty$), so we must have $\alpha + \beta > 0$, $\alpha + \frac{\beta}{1+\gamma} > 0$, and $\alpha + \beta(1-\gamma) \geq 0$. Then, applying the ratio test to the sum

$$\sum_{k\geq 0} p_k \;=\; p_0 \sum_{k=0}^{\infty} \prod_{n=0}^{k-1} \left(\alpha + \beta\,\frac{1-\gamma}{1-\gamma^{n+1}}\right)$$

we see that it converges iff $0 \le \alpha + \beta(1-\gamma) < 1$. Thus a necessary and sufficient condition for a solution of (3.1) with infinite support (random variable with finite support cannot be infinitely divisible) to exist is that

$$\min\left\{\alpha+\beta,\; \alpha + \frac{\beta}{1+\gamma}\right\} > 0 \qquad \text{and} \qquad 0 \le \alpha + \beta(1-\gamma) < 1 \;.$$

Rewriting (3.1) as

$$(3.2) \qquad (1-\gamma^{n+1})\, f_{N_{\alpha,\beta,\gamma}}(n+1) \;=\; \big[\alpha+\beta(1-\gamma)\big]\, f_{N_{\alpha,\beta,\gamma}}(n) - \alpha\,\gamma^{n+1} f_{N_{\alpha,\beta,\gamma}}(n)\;,$$

for $\gamma \in (-1,\,1)$, $n = 0, 1, ...$, multiplying both sides by s^{n+1} and summing we get

$$(3.3) \qquad \Big[1 - \big(\alpha+\beta(1-\gamma)\big)\, s\Big]\, \mathcal{G}_{\alpha,\beta,\gamma}(s) \;=\; (1-\alpha\,\gamma\, s)\, \mathcal{G}_{\alpha,\beta,\gamma}(\gamma s)\;,$$

where $\mathcal{G}_{\alpha,\beta,\gamma}(s) = \sum\limits_{n=0}^{\infty} f_{N_{\alpha,\beta,\gamma}}(n)\, s^n$ denotes the probability generating function of the probability mass function $\big\{f_{N_{\alpha,\beta,\gamma}}(n)\big\}_{n=0}^{\infty}$, and from that

$$(3.4) \qquad \mathcal{G}_{\alpha,\beta,\gamma}(s) \;=\; \mathcal{G}_{\alpha,\beta,\gamma}(\gamma^{n+1} s) \prod_{k=0}^{n} \frac{1-\alpha\,\gamma^{k+1} s}{1 - \big[\alpha+\beta(1-\gamma)\big]\,\gamma^k s}\;.$$

Observing that

$$(3.5) \qquad \frac{\mathcal{G}_{\alpha,\beta,\gamma}(s)}{\mathcal{G}_{\alpha,\beta,\gamma}(1)} \;=\; \frac{\mathcal{G}_{\alpha,\beta,\gamma}(\gamma^{n+1} s)}{\mathcal{G}_{\alpha,\beta,\gamma}(\gamma^{n+1})} \prod_{k=0}^{n} \frac{\dfrac{1-\alpha\,\gamma^{k+1} s}{1-\big[\alpha+\beta(1-\gamma)\big]\,\gamma^k s}}{\dfrac{1-\alpha\,\gamma^{k+1}}{1-\big[\alpha+\beta(1-\gamma)\big]\,\gamma^k}}$$

and letting $n \to \infty$,

$$(3.6) \qquad \mathcal{G}_{\alpha,\beta,\gamma}(s) \;=\; \prod_{k=0}^{\infty} \frac{1-\alpha\,\gamma^{k+1} s}{1-\alpha\,\gamma^{k+1}}\;\frac{1-\big[\alpha+\beta(1-\gamma)\big]\,\gamma^k}{1-\big[\alpha+\beta(1-\gamma)\big]\,\gamma^k s}\;.$$

If $\gamma \in [0,1)$, $\alpha < 0$ and $\beta \in (-\frac{\alpha}{1-\gamma},\, \frac{1-\alpha}{1-\gamma})$, we recognize in

$$\mathcal{G}_{\alpha,\beta,\gamma}(s) \;=\; \prod_{k=0}^{\infty} \frac{1-\alpha\,\gamma^{k+1} s}{1-\alpha\,\gamma^{k+1}}\;\frac{1-\big[\alpha+\beta(1-\gamma)\big]\,\gamma^k}{1-\big[\alpha+\beta(1-\gamma)\big]\,\gamma^k s}\;,$$

the probability generating function of an infinite sum of independent random variables, the k-th summand being the result of randomly adding 1, with probability $\frac{\alpha\,\gamma^{k+1}}{\alpha\,\gamma^{k+1}-1}$, to an independent *Geometric*$(1-[\alpha+\beta(1-\gamma)]\gamma^k)$ random variable.

The limiting case $\gamma = 1$ may be approached as follows: rewriting (3.3) as

$$\frac{\mathcal{G}_{\alpha,\beta,\gamma}(s) - \mathcal{G}_{\alpha,\beta,\gamma}(\gamma s)}{\alpha\, s\Big[\mathcal{G}_{\alpha,\beta,\gamma}(s) - \mathcal{G}_{\alpha,\beta,\gamma}(\gamma s)\Big] + (1-\gamma)\, s\Big[\beta\, \mathcal{G}_{\alpha,\beta,\gamma}(s) + \alpha\, \mathcal{G}_{\alpha,\beta,\gamma}(\gamma s)\Big]} = 1 \;,$$

dividing the numerator and the denominator by $(1-\gamma)s$ and letting $\gamma \to 1$, we get

$$\frac{\mathcal{G}'_{\alpha,\beta,1}(s)}{\alpha\, s\, \mathcal{G}'_{\alpha,\beta,1}(s) + \beta\, \mathcal{G}_{\alpha,\beta,1}(s) + \alpha\, \mathcal{G}_{\alpha,\beta,1}(s)} = 1 \iff \frac{\mathcal{G}'_{\alpha,\beta,1}(s)}{\mathcal{G}_{\alpha,\beta,1}(s)} = \frac{\alpha+\beta}{1-\alpha\, s} \;,$$

the expression we obtain working out the probability generating function in Panjer's iterative expression $p_{\alpha,\beta}(n+1) = (\alpha + \frac{\beta}{n+1})\, p_{\alpha,\beta}(n),\ \alpha, \beta \in \mathbb{R},\ n = 0, 1, \ldots$.

We now focus on the case $\alpha = 0$, for which $\beta \in (0, \frac{1}{1-\gamma})$, and

$$\mathcal{G}_{0,\beta,\gamma}(s) = \prod_{k=0}^{\infty} \frac{1 - \beta(1-\gamma)\, \gamma^k}{1 - \beta(1-\gamma)\, \gamma^k s} = \prod_{k=0}^{\infty} \frac{1 - w_k}{1 - w_k s} \;, \tag{3.7}$$

where $w_k = \beta(1-\gamma)\gamma^k$. If $\gamma \in [0, 1)$, we get that have $N_{0,\beta,\gamma} = \sum_{k=0}^{\infty} W_k$, with $W_k \frown$ *Geometric*$(1 - \beta(1-\gamma)\, \gamma^k)$ independent summands. If $\gamma = 0$, the above expression simplifies to $\mathcal{G}_{0,\beta,0}(s) = \frac{1-\beta}{1-\beta s}$. Therefore we conclude that $N_{0,\beta,0} = N_{\beta,0} \frown$ *Geometric*$(1-\beta)$, $\beta \in (0, 1)$.

Let us point out that the probability mass function of a random variable $N_{0,\beta,\gamma}$, $\gamma \in (-1, 1)$, trivially satisfies

$$\frac{1-\gamma^{n+1}}{1-\gamma}\, p_{n+1} = \sum_{k=0}^{n} p_k\, r_{n-k} \;,$$

with $r_0 = \beta$ and $r_1 = r_2 = \cdots = r_n = 0$, provided that

$$0 < \beta = \sum_{n=0}^{\infty} r_n\, s^n = \mathcal{H}(s) < \frac{1}{1-\gamma} \;,$$

a point which will be of relevance in the following section.

4. DISCRETE INFINITELY DIVISIBLE DISTRIBUTIONS AND $\mathcal{C}_\gamma$ CLASSES

In what follows we investigate the classes $\mathcal{C}_\gamma$, $\gamma \in (-1, 1)$, of nondegenerate counting random variables (distributions, p.g.f.) whose probability mass function satisfies $\tilde{p}_0 > 0$ and the general recursive relation

$$\frac{1-\gamma^{n+1}}{1-\gamma}\, \tilde{p}_{n+1} = \sum_{k=0}^{n} \tilde{p}_k\, r_{n-k} \;, \qquad n = 0, 1, \ldots \;, \tag{4.1}$$

with $r_k \geq 0$, which extends Panjer's recursive expression for the probability mass function of the classes of Poisson stopped sums ($\mathcal{C}_1$) and of geometric stopped sums ($\mathcal{C}_0$). It is well known that any geometric infinitely divisible lattice distribution is infinitely divisible in the classical sense, a result that follows from the fact that $\frac{1-p}{1-ps} = \exp\{\ln(1-p)\,[\mathcal{P}(s)-1]\}$, where $\mathcal{P}(s) = -\frac{1}{\ln(1-p)} \sum_{k=0}^{\infty} \frac{(ps)^k}{k}$ is the p.g.f. of a logarithmic random variable.

As before, multiplying both members of (4.1) by s^{n+1} and summing for $n \geq 0$, we obtain

$$\frac{\mathcal{G}(s)-\mathcal{G}(\gamma s)}{1-\gamma} = s\,\mathcal{G}(s)\,\mathcal{H}_\gamma(s)\;, \tag{4.2}$$

where $\mathcal{G}(s) = \sum_{n=0}^{\infty} \tilde{p}_n s^n$ and $\mathcal{H}_\gamma(s) = \sum_{n=0}^{\infty} r_n s^n$ converges at least for $|s| \leq 1$. Thus $\mathcal{H}_\gamma$ is by definition abs. mon. Since we have excluded degenerate solutions to (4.1), we must have $\mathcal{H}_\gamma(0) = r_0 = \frac{\tilde{p}_1}{\tilde{p}_0} > 0$.

If $\gamma \in [0,\,1)$, we have

$$\begin{aligned} 1 \;\geq\; \sum_{n=0}^{\infty} \tilde{p}_{n+1} &= \sum_{n=0}^{\infty} \frac{1-\gamma}{1-\gamma^{n+1}} \sum_{k=0}^{n} \tilde{p}_k r_{n-k} \\ &= \sum_{k=0}^{\infty} \tilde{p}_k \sum_{n=0}^{\infty} \frac{(1-\gamma)\,r_n}{1-\gamma^{n+k+1}} \\ &> \sum_{k=0}^{\infty} \tilde{p}_k \sum_{n=0}^{\infty} (1-\gamma)\,r_n \;=\; (1-\gamma) \sum_{n=0}^{\infty} r_n\;, \end{aligned}$$

and therefore $|\mathcal{H}_\gamma(s)| \leq \mathcal{H}_\gamma(1) = \sum_{n=0}^{\infty} r_n < \frac{1}{1-\gamma}$ for $|s| \leq 1$.

If $\gamma \in (-1,\,0)$, then $\frac{1-\gamma^{i+1}}{1-\gamma} \leq 1$ for $i = 0, 1, ...$, and by a similar reasoning we conclude that in this case $|\mathcal{H}_\gamma(s)| < 1$ for $|s| \leq 1$.

As was seen in the previous section, the p.m.f. of $N_{0,\beta,\gamma}$ verifies recursion (4.1) with $r_0 = \beta$ and $r_1 = r_2 = ... = 0$, with $0 < \beta < \frac{1}{1-\gamma}$.

We have the following result:

Theorem 4.1. *Let W be a random variable with p.g.f. $\mathcal{G}$, and $\gamma \in (-1, 1)$.*

$$W \in \mathcal{C}_\gamma \qquad \text{iff} \;\; \mathcal{G}(s) = \prod_{k=0}^{\infty} \frac{1-(1-\gamma)\,\gamma^k\,\mathcal{H}_\gamma(\gamma^k)}{1-(1-\gamma)\,\gamma^k s\,\mathcal{H}_\gamma(\gamma^k s)}\;,$$

where $\mathcal{H}_\gamma$ is a unique abs. mon. function such that $\mathcal{H}_\gamma(0) > 0$ and $\mathcal{H}_\gamma(1) <$ $\max\{1, \frac{1}{1-\gamma}\}$.

Thus, if $\gamma \in [0,1)$ the elements of $\mathcal{C}_\gamma$ are infinite sums $W = \sum_{k=0}^{\infty} X_k$ of independent geometric stopped sums $X_k = \sum_{i=1}^{N_k} Y_{ki}$, whose subordinators are $N_k \frown \textit{Geometric}(1-(1-\gamma)\,\gamma^k\,\mathcal{H}_\gamma(\gamma^k))$ random variables, and whose i.i.d. summands $Y_{ki} \overset{\mathrm{d}}{=} Y_k$ have the p.g.f. $\mathcal{P}_k(s) = \frac{s\,\mathcal{H}_\gamma(\gamma^k s)}{\mathcal{H}_\gamma(\gamma^k)}$.

Proof: We have established that

$$\frac{\mathcal{G}(s)-\mathcal{G}(\gamma s)}{1-\gamma} = s\,\mathcal{G}(s)\,\mathcal{H}_\gamma(s) \quad \Longleftrightarrow \quad \frac{\mathcal{G}(s)}{\mathcal{G}(\gamma s)} = \frac{1}{1-(1-\gamma)\,s\;\mathcal{H}_\gamma(s)}\;.$$

Iterating the above expression, similarly to what we have done to obtain (3.6), we finally get

$$\mathcal{G}(s) \;=\; \prod_{k=0}^{\infty} \frac{1-(1-\gamma)\,\gamma^k\mathcal{H}_\gamma(\gamma^k)}{1-(1-\gamma)\,\gamma^k s\,\mathcal{H}_\gamma(\gamma^k s)}\;. \tag{4.3}$$

If $\gamma \in [0,1)$, we further have

$$\mathcal{G}(s) \;=\; \prod_{k=0}^{\infty} \frac{1-w_k}{1-w_k\,\frac{s\,\mathcal{H}_\gamma(\gamma^k s)}{\mathcal{H}_\gamma(\gamma^k)}} \;=\; \prod_{k=0}^{\infty} \frac{1-w_k}{1-w_k\,\mathcal{P}_k(s)}$$

where $w_k = (1-\gamma)\,\gamma^k\mathcal{H}_\gamma(\gamma^k)$, and the $\mathcal{P}_k(s) = \frac{s\,\mathcal{H}_\gamma(\gamma^k s)}{\mathcal{H}_\gamma(\gamma^k)}$ are (unique) probability generating functions such that $\mathcal{P}_k(0) = 0$. □

Theorem 4.2. *Let W be a counting random variable with p.g.f. $\mathcal{G}$, and $\gamma \in (-1,1)$. $W \in \mathcal{C}_\gamma$ iff $\mathcal{H}_\gamma(s) = \frac{\mathcal{G}(s)-\mathcal{G}(\gamma s)}{(1-\gamma)\,s\,\mathcal{G}(s)}$ is abs. mon.*

We can use this result to show that the geometric distribution verifies (4.1) for nonnegative γ. In fact, if $X_\theta \frown \textit{Geometric}(1-\theta)$, with $0<\theta<1$, we have $r_k = \gamma^k\theta^{k+1} \geq 0$ and $\mathcal{H}_\gamma(1) = \frac{\theta}{1-\gamma\theta} < \frac{1}{1-\gamma}$. Given the uniqueness of the coefficients of $\mathcal{H}_\gamma$, we may also conclude that the geometric distribution does not belong to $\mathcal{C}_\gamma$ when $\gamma \in (-1,0)$.

The truncated geometric distribution with support on the even integers, Y_θ, given by the p.m.f.

$$p_n = \begin{cases} (1-\theta^2)\,\theta^n & \text{if } n = 2k \text{ even} \\ 0 & \text{if } n = 2k+1 \text{ odd} \end{cases}\;, \qquad 0<\theta<1\;,$$

is an element of $\mathcal{C}_\gamma$ for all $\gamma \in (-1,1]$, since it verifies (4.1) with $r_{2k} = 0$, $r_{2k+1} = (1+\gamma)\;\gamma^{2k}\,\theta^{2k+2}$, and $\mathcal{H}_\gamma(1) = \frac{(1+\gamma)\theta^2}{1-(\gamma\theta)^2} < \frac{1}{1-\gamma}$.

It's interesting to note that the p.g.f. of Y_θ is $\mathcal{G}_{Y_\theta}(s) = \frac{1-\theta^2}{1-\theta^2 s^2} = \mathcal{G}_{X_{\theta^2}}(s^2)$. It is not difficult to show that if $X \in \mathcal{C}_0$ has the p.g.f. $\mathcal{G}$, then $\mathcal{G}(s^2)$ is the p.g.f. of an element of $\mathcal{C}_\gamma$, for every $\gamma \in (-1,1]$.

Corolary 4.2.1. Let W be a counting random variable with p.g.f. $\mathcal{G}$, and $\gamma \in (-1,1)$. If $W \in \mathcal{C}_\gamma$, $\frac{\mathcal{G}(s)}{\mathcal{G}(\gamma s)}$ is absolutely monotone.

Proof: From the proof of Theorem 4.1, $\frac{\mathcal{G}(s)}{\mathcal{G}(\gamma s)} = \frac{1}{1-(1-\gamma)\, s\, \mathcal{H}_\gamma(s)}$. If $\gamma \in [0,1)$, we have $s\,\mathcal{H}_\gamma(s) \le \mathcal{H}_\gamma(1) < \frac{1}{1-\gamma}$ for $0 \le s \le 1$; on the other hand, if $\gamma \in (-1,0)$ we have $(1-\gamma) s\,\mathcal{H}_\gamma(s) \le (1-\gamma)\,\mathcal{H}_\gamma(1) < 1$ for $0 \le s \le \frac{1}{1-\gamma}$. Thus, it follows from property 3(b) of abs. mon. functions that $\frac{\mathcal{G}(s)}{\mathcal{G}(\gamma s)}$ is abs. mon. (in $[0,1]$ for nonnegative γ, and in $[0, \frac{1}{1-\gamma}]$ for negative γ). □

Corolary 4.2.2. For $\gamma \in (-1,1)$, $\mathcal{C}_\gamma \subset \mathcal{C}_1$.

Proof: Taking derivatives on both sides of $1 - \frac{\mathcal{G}(\gamma s)}{\mathcal{G}(s)} = (1-\gamma)\, s\, \mathcal{H}_\gamma(s)$, we obtain

$$\frac{\mathcal{G}'(s)\,\mathcal{G}(\gamma s) - \gamma\,\mathcal{G}'(\gamma s)\,\mathcal{G}(s)}{\mathcal{G}^2(s)} = (1-\gamma)\,\frac{\mathrm{d}}{\mathrm{d}s}\big[s\,\mathcal{H}_\gamma(s)\big] ,$$

equivalent to

$$\frac{\mathcal{G}'(s)}{\mathcal{G}(s)} - \gamma\,\frac{\mathcal{G}'(\gamma s)}{\mathcal{G}(\gamma s)} = (1-\gamma)\,\frac{\mathcal{G}(s)}{\mathcal{G}(\gamma s)}\,\frac{\mathrm{d}}{\mathrm{d}s}\big[s\,\mathcal{H}_\gamma(s)\big] . \tag{4.4}$$

Therefore, in view of Corollary 4.2.1 and of property 1 of abs. mon. functions $\frac{\mathcal{G}'(s)}{\mathcal{G}(s)} - \gamma\,\frac{\mathcal{G}'(\gamma s)}{\mathcal{G}(\gamma s)}$ is abs. mon. which in turn (property 2 of abs. mon. functions) implies that $\frac{\mathcal{G}'(s)}{\mathcal{G}(s)}$ is abs. mon.

The result follows from Corollary 2.1.1. □

The inclusion is strict: the *Poisson*(μ) distribution belongs to $\mathcal{C}_1$ for all $\mu > 0$, but does not belong to $\mathcal{C}_\gamma$ when $\gamma \in (-1,1)$, since from Theorem 4.2 we have $r_k = (-1)^k (1-\gamma)^k \frac{\mu^{k+1}}{(k+1)!}$, so that $\mathcal{H}_\gamma$ is not abs. mon.

Corolary 4.2.3. For $|\gamma_1| \le \gamma_2 < 1$, $\mathcal{C}_{\gamma_1} \subset \mathcal{C}_{\gamma_2}$.

Proof: Let $\mathcal{G}$ be the p.g.f. of a random variable $W \in \mathcal{C}_{\gamma_1} \subset \mathcal{C}_1$.

$$\frac{\mathcal{H}_{\gamma_2}(s) - \gamma_1 \mathcal{H}_{\gamma_2}(\gamma_1 s)}{\mathcal{H}_{\gamma_1}(s) - \gamma_2 \mathcal{H}_{\gamma_1}(\gamma_2 s)} = \frac{1-\gamma_1}{1-\gamma_2}\,\frac{\frac{\mathcal{G}(\gamma_1\gamma_2 s)}{\mathcal{G}(\gamma_1 s)} - \frac{\mathcal{G}(\gamma_2 s)}{\mathcal{G}(s)}}{\frac{\mathcal{G}(\gamma_1\gamma_2 s)}{\mathcal{G}(\gamma_2 s)} - \frac{\mathcal{G}(\gamma_1 s)}{\mathcal{G}(s)}} = \frac{1-\gamma_1}{1-\gamma_2}\,\frac{\mathcal{G}(\gamma_2 s)}{\mathcal{G}(\gamma_1 s)} .$$

From Corollary 2.1.1.(3), $\frac{\mathcal{G}(\gamma_2\, s)}{\mathcal{G}(\gamma_1 s)}$ is abs. mon, and from property 2 of abs. mon. functions $\mathcal{H}_{\gamma_1}(s)-\gamma_2\,\mathcal{H}_{\gamma_1}(\gamma_2\, s)$ is abs. mon. Then $\mathcal{H}_{\gamma_2}(s)-\gamma_1\mathcal{H}_{\gamma_2}(\gamma_1\, s)$, and therefore $\mathcal{H}_{\gamma_2}$, are also abs. mon., which proves that $W\in\mathcal{C}_{\gamma_2}$. □

We can see that the inclusion is strict directly from (4.1). Suppose that $-1<\gamma<\eta<1$ and $0<\beta<\frac{1}{1-\eta}$. We know that $N_{0,\,\beta,\,\eta}\in\mathcal{C}_\eta$, since its p.m.f. satisfies $\frac{1-\eta^{n+1}}{1-\eta}\,p_{n+1}=\beta\,p_n$. Assume that $N_{0,\,\beta,\,\eta}\in\mathcal{C}_\gamma$, that is, $\frac{1-\gamma^{n+1}}{1-\gamma}\,p_{n+1}=\sum_{k=0}^{n}p_k r_{n-k}$. Then $p_1=p_0 r_0=\beta\,p_0$ implies $r_0=\beta$, and

$$(4.5)\qquad (1+\gamma)\,p_2 \;=\; \frac{\beta}{1+\eta}\,p_1(1+\eta+\gamma-\eta) \;=\; p_1 r_0+p_0 r_1$$

implies $r_1=-\,\frac{\eta-\gamma}{1+\eta}\,\beta^2$. But this is negative, therefore $N_{0,\,\beta,\,\eta}\notin\mathcal{C}_\gamma$.

Corolary 4.2.4. Let W be a counting random variable with p.g.f. $\mathcal{G}$, W_γ the random variable with p.g.f. $\mathcal{G}_\gamma(s)=\frac{\mathcal{G}(\gamma)\,\mathcal{G}(s)}{\mathcal{G}(\gamma\, s)}$, and $\gamma\in(-1,1)$. $W\in\mathcal{C}_\gamma$ iff $W_\gamma\in\mathcal{C}_0$.

Proof: As $W\in\mathcal{C}_\gamma \implies W\in\mathcal{C}_1$, from Corollary 2.1.1 we know that $\mathcal{G}_\gamma(s)=\frac{\mathcal{G}(\gamma)\,\mathcal{G}(s)}{\mathcal{G}(\gamma\, s)}$ is a p.g.f.

From the proof of Theorem 2.2 (or simply by taking $\gamma=0$ in Theorem 4.1), in what concerns this p.g.f. $\mathcal{G}_\gamma$ we obtain, with self-explaining notations,

$$(4.6)\qquad \mathcal{H}_0^{(\mathcal{G}_\gamma)}(s) \;=\; \frac{\mathcal{G}_\gamma(s)-\mathcal{G}_\gamma(0)}{s\,\mathcal{G}_\gamma(s)} \;=\; \frac{\mathcal{G}(s)-\mathcal{G}(\gamma s)}{s\,\mathcal{G}(s)} \;=\; (1-\gamma)\,\mathcal{H}_\gamma^{(\mathcal{G})}(s)$$

and therefore $\mathcal{H}_0^{(\mathcal{G}_\gamma)}$ is abs. mon. iff $\mathcal{H}_\gamma^{(\mathcal{G})}$ is abs. mon. □

5. FURTHER COMMENTS

1. Geometric infinite divisibility arose from Kovalenko's (1965) extensions of Rényi's (1956) work on random rarefaction, with the general characterization of geometric stable laws given in Kozubowski (1994). This led to a general definition of $\mathcal{N}$-summation schemes, the classical summation scheme being the special case $N_p=\frac{1}{p}$ (degenerate random variables, and therefore a non-random sum of random variables). It is well known that for some families $\mathcal{N}=\{N_p,\ p\in(0,1),\ \mathbb{E}(N_p)=\frac{1}{p}\}$ there exists $\mathcal{N}$-Gaussian laws (for instance for $N_p\frown$ *Geometric*(p), the corresponding $\mathcal{N}$-Gaussian random variables being the Laplace random variables), while other N_p, for instance $N_p\frown$ *Poisson*$(\frac{1}{p})$, do not admit $\mathcal{N}$-Gaussian laws. Although it is easy to prove that in more general branching setings $\mathcal{N}$-Gaussian laws do exist, only the usual Gaussian law and the Laplace geometric–Gaussian law are explicitly exhibited in the references we know.

This research arose from the observation that $\mathcal{C}_0 \subset \mathcal{C}_1$ and that, more generaly, $0 < \gamma_1 < \gamma_2 < 1 \implies \mathcal{C}_0 \subset \mathcal{C}_{\gamma_1} \subset \mathcal{C}_{\gamma_2} \subset \mathcal{C}_1$.

Our aim was either to prove that there exist $\gamma \in (0,1)$ such that for $\gamma_1 \leq \gamma$ we could exhibit a $\mathcal{N}$-Gaussian law in $\mathcal{C}_{\gamma_1}$ — which we couldn't — or else to extend $\mathcal{C}_\gamma$ classes for $\gamma < 0$ — which we did — and show that for those it was possible to construct $\mathcal{N}$-Gaussian random variables. Unfortunately for $-1 < \gamma_1 < \gamma_2 < 0$ the chain of inclusions $\mathcal{C}_{\gamma_1} \subset \mathcal{C}_{\gamma_2} \subset \mathcal{C}_0$ is no longer valid.

2. The extension of Katz–Panjer's iterative relation

$$\frac{f(n+1)}{f(n)} = \alpha + \frac{\beta}{n+1} = \alpha + \beta\, \mathbb{E}(U_0^n)\ , \qquad n = 0,1,..., \quad \alpha,\beta \in \mathbb{R}\ , \tag{5.1}$$

by

$$\frac{f(n+1)}{f(n)} = \alpha + \beta\, \frac{\mathbb{E}(U_0^n)}{\mathbb{E}(U_\gamma^n)}\ , \qquad n = 0,1,..., \quad \alpha,\beta \in \mathbb{R}\ , \tag{5.2}$$

where $U_\gamma \frown$ *Uniform*$(\gamma,1)$, $\gamma \in (-1,1]$ may seem arbitrary at this stage, unless it is considered as a first step in extending (5.1) by using more general *Beta*, of which the *Uniform* in (5.2) isn't but a special case, or even more general random variables. Naturaly $\{f(n)\}_{n\in\mathbb{N}}$ is not a p.m.f. unless the restrictions in the parameters are very strong.

3. Panjer's class $\Pi = \Pi^{(0)}$ has been generalized by Sundt and Jewell (1981), who considered the class $\Pi^{(1)}$ of discrete random variables whose probability mass function satisfies

$$f_{\alpha,\beta}(n+1) = \left(\alpha + \frac{\beta}{n+1}\right) f_{\alpha,\beta}(n)\ , \qquad \alpha,\beta \in \mathbb{R}, \quad n = 1,2,... \tag{5.3}$$

Willmot (1987) published the definitive characterization of $\Pi^{(1)}$: the probability mass function of a discrete random variable N, with support $\mathcal{S} = \{1,2,...\}$, satisfies the above expression if N is either a zero-truncated Binomial, Poisson or Negative Binomial random variable, or a Logarithmic (when $\alpha \in (0,1)$ and the index $\frac{\alpha}{\alpha+\beta} \to 0$) or an Engen (1974) Extended Negative Binomial random variable (index $\frac{\alpha}{\alpha+\beta} \in (-1,0)$, where $\alpha \in (0,1]$), and general solutions N^*, with support $\mathcal{S} = \{0,1,2,...\}$, arise from a *hurdle process* (Cameron and Trivedi, 1998, pp. 123–125) $N^* = \begin{cases} 0 & N \\ p_0 & 1-p_0 \end{cases}$, where N is one of the above variables. Klugman *et al.* (1998) describe the solutions as *zero modified* N variables.

Hess, Lewald and Schmidt (2002) considered the even more general setting $\Pi^{(k)}$, $k = 0, 1, ...$, in which the probability mass functions satisfy

$$f_{\alpha,\beta}(n+1) = \left(\alpha + \frac{\beta}{n+1}\right) f_{\alpha,\beta}(n) , \qquad \alpha, \beta \in \mathbb{R}, \quad n = k, k+1, ... , \tag{5.4}$$

giving a complete description of $\Pi^{(k)}$, $k = 0, 1, ...$ in terms of $\{0, 1..., k-1\}$ modified basic claim number distributions, i.e., the left k-truncated binomial, Poisson, and negative binomial distributions, the other basic claim number distributions are the left truncated *Logarithmic*(k, θ) distribution, and the left truncated *Engen*(k, β, θ) distribution. The extension

$$\frac{f_{\alpha,\beta}(n+1)}{f_{\alpha,\beta}(n)} = \alpha + \beta\,\frac{1-\gamma}{1-\gamma^{n+1}} , \qquad \alpha, \beta \in \mathbb{R}, \quad n = k, k+1, ... , \tag{5.5}$$

of (3.1) may investigated along similar lines, but with very cumbersome results.

REFERENCES

[1] Bernstein, S. (1928). Sur les fonctions absolument monotones, *Acta Mathematica*, **51**, 1–66.

[2] Cameron, A.C. and Trivedi, P.K. (1998). *Regression Analysis of Count Data*, Cambridge University Press, Cambridge.

[3] Dhaene, J. and Sundt, B. (1998). On approximating distributions by approximating their De Pril transforms, *Scand. Actuar. J.*, 1–23.

[4] Engen, S. (1974). On species frequency models, *Biometrika*, **61**, 263–270.

[5] Feller, W. (1968). *An Introduction to Probability Theory and its Applications*, vol. I, Wiley, New York.

[6] Gurland, J. (1957). Some interrelations among compound and generalized distributions, *Biometrika*, **44**, 265–268.

[7] Hess, K.Th.; Lewald, A. and Schmidt, K.D. (2002). An extension of Panjer's recursion, *ASTIN Bulletin*, **32**, 283–297.

[8] Johnson, N.L.; Kotz, S. and Kemp, A.W. (1992). *Univariate Discrete Distributions*, Wiley, New York.

[9] Katz, L. (1965). *Unified treatment of a broad class of discrete probability distributions.* In "Classical and Contagious Discrete Distributions", Pergamon Press, Oxford, 175–182.

[10] Klugman, S.A.; Panjer, H.H. and Willmot, G.E. (1998). *Loss Models: From Data to Decisions*, Wiley, New York.

[11] Kovalenko, I.N. (1965). On a class of limit distributions for rarefied flows of homogeneous events, *Lit. Mat. Sbornik*, **5**, 569–573. (*Selected Transl. Math. Statist. and Prob.* **9**, Providence, Rhode Island, 1971, 75–81.)

[12] Kozubowski, T.J. (1994). Representation and properties of geometric stable laws, *Approximation, Probability, and Related Fields*, Plenum, New York, 321–337.

[13] Ospina, A.V. and Gerber, H.U. (1987). A simple proof of Feller's characterization of the compound Poisson distribution, *Insurance: Mathematics and Economics*, **6**, 63–64.

[14] Panjer, H.H. (1981). Recursive evaluation of a family of compound distributions, *ASTIN Bulletin*, **12**, 22–26.

[15] Rényi, A. (1956). A characterization of the Poisson process, *MTA Mat. Kut. Int. Közl.*, **1**, 519–527. (English translation: *Selected Papers of Alfred Rényi,* **1**, *1948–1956*, P. Turán, ed., 622–279, Akadémiai Kiadó, Budapest, with a note by D. Szász on ulterior developments up to 1976).

[16] Rólski, T.; Schmidli, H.; Schmidt, V. and Teugels, J. (1999). *Stochastic Processes for Insurance and Finance*, Wiley, New York.

[17] Skellam, J.G. and Shenton, L.R. (1957). Distributions associated with random walks and recurrent events (with discussion), *J. Roy. Statist. Soc.*, **B 19**, 64–118.

[18] Srivastava, H.M. and Manocha, H.L. (1984). *A Treatise on Generating Functions*, Horwood, Chichester.

[19] Sundt, B. and Jewell, W.S. (1981). Further results on recursive evaluation of compound distributions, *ASTIN Bulletin*, **12**, 27–39.

[20] Widder, D.V. (1946). *The Laplace Transform*, Princeton Univ. Press, Princeton.

[21] Willmot, G.E. (1987). Sundt and Jewell's family of discrete distributions, *ASTIN Bulletin*, **18**, 17–29.