REVSTAT – Statistical Journal
Volume 4, Number 3, November 2006, 249–273

EXTREMES OF PERIODIC INTEGER-VALUED SEQUENCES WITH EXPONENTIAL TYPE TAILS

Authors: Andreia Hall
– Departamento de Matemática, UI&D Matemática e Aplicações, Universidade de Aveiro, Portugal
andreia@mat.ua.pt

Manuel G. Scotto
– Departamento de Matemática, UI&D Matemática e Aplicações, Universidade de Aveiro, Portugal
mscotto@mat.ua.pt

Received: July 2006 Accepted: October 2006

Abstract:

- This paper aims to analyze the extremal properties of periodic integer-valued sequences with marginal distribution belonging to a particular class defined by Anderson [1970. J. Appl. Probab. 7, 99–113] where the tail decays exponentially. An expression for calculating the extremal index of sequences satisfying certain local conditions, similar to those introduced by Chernick *et al.* [1991. Adv. Appl. Prob. 6, 711–731] is obtained. An application to infinite moving averages and max-autoregressive sequences is included. These results generalize the ones obtained for the stationary case.

Key-Words:

- *extreme value theory; binomial thinning; periodic sequences.*

AMS Subject Classification:

- 60G70, 60G10.

1. INTRODUCTION

The analysis of integer-valued time series has become an important area of research in the last two decades partially because its wide applicability to experimental biology (Zhou and Basawa [34]), social science (McCabe and Martin [24]), international tourism demand (Nordström [29], Garcia-Ferrer and Queralt [16], Brännäs *et al.* [12]), queueing systems (Ahn *et al.* [7]) and economy (Quoreshi [30]). We refer to McKenzie [28] for an overview of the early work in this area. Among the most successful integer-valued time series models proposed in the literature we mention the INAR(p) model and the INMA(q) model. The former was first introduced by McKenzie (e.g., [26]) and Al-Osh and Alzaid [1] for the case $p=1$. Empirical relevant extensions have been suggested by Brännäs ([9], explanatory variables), Blundell *et al.* ([8], panel data), Brännäs and Hellström ([11], extended dependence structure), and more recently by Silva *et al.* ([32], replicated data). Extensions and generalizations were introduced by Du and Li [14] and Latour [22]. The INMA(q) model was proposed by Al-Osh and Alzaid [2] and subsequently studied by Brännäs and Hall [10]. Related models were introduced by Aly and Bouzar ([4], [5]) and Zhu and Joe [35].

Within the reasonably large spectrum of integer-valued models proposed in the literature, little is known about its extremal properties. Anderson [6] gave a noticeable contribution to the study of the extremal properties of integer-valued independent and identically distributed (i.i.d.) sequences and as an example of application, the author analyzed the behavior of the maximum queue length for $M/M/1$ queues. Extensions of Anderson's results were proposed by Hooghiemstra *et al.* [21] who provide bounds and approximations for the distribution of the maximum queue length for $M/M/s$ queues, based on an asymptotic analysis involving the extremal index. McCormick and Park [25] were the first to study the extremal properties of some models obtained as discrete analogues of continuous models, replacing scalar multiplication by *random thinning*. Hall [17] analyzed the asymptotic behavior of the maximum term of a particular Markovian model. [18] provided results regarding the limiting distribution of the maximum of sequences within a generalized class of integer-valued moving averages driven by i.i.d. heavy-tailed innovations. Extensions for exponential type-tailed innovations have been studied by Hall [19]. More recently, Hall and Moreira [20] derived the extremal properties of a particular moving average count data model introduced by McKenzie [27].

It is worth to mention that all the references given in the previous paragraph deal with the case of stationary sequences. In contrast, however, the study of the extremal properties of integer-valued non-stationary sequences has been overlooked in the literature. This paper aims at giving a contribution towards this direction. In particular we consider periodic sequences with marginal distributions within a particular class of discrete distributions first considered by Anderson [6]. Potential applications can be found in the analysis of the number of hotel guest nights where the series exhibit strong seasonal pattern with a peak in July–August and a trough in December–February, and in the study of the number of claims of short-term disability benefits made by injured workers since it is expected to see fewer claims in the winter months and more in the summer months.

The term periodic is used in this paper in a different sense than in the literature of periodic stochastic processes in which a sequence $(X_n)_{n\in\mathbb{N}}$ is said to be periodically stationary (in the wide sense) if its mean and covariance structure are periodic functions of time with the same period. This class of processes, however, does not appear to be sufficiently flexible to deal with data which exhibit non-standard features like nonlinearity and/or heavy tails. In this paper by periodic sequence, with period say T, we mean that for a sequence of random variables (rv's) $(X_n)_{n\in\mathbb{N}}$ there exist an integer $T \geq 1$ such that, for each choice of integers $1 \leq i_1 < i_2 < \cdots < i_n$, $(X_{i_1}, ..., X_{i_n})$ and $(X_{i_1+T}, ..., X_{i_n+T})$ are identically distributed. The period T will be considered the smallest integer satisfying the above definition.

The rest of the paper is organized as follows: Section 2 provides the necessary theoretical background; Section 3 includes the main result that leads to the calculation of the limiting distribution of the maximum term; in Section 4 the previous results are applied to a particular class of max-autoregressive sequences generalizing the results of Hall [17]; finally, in Section 5 we look at the distribution of the maximum term of periodic moving average sequences obtained as discrete analogues of classical moving averages with periodic (but independent) innovations, generalizing the results given in Hall [19].

In this paper we want to highlight the following issues:

a) Under fairly general dependence conditions, integer-valued T-periodic sequences with marginal distribution in Anderson's class exhibit a quasi-stable non-degenerate limiting distribution of the maximum term which is obtained as a generalization of the stationary case.

b) The expression of the extremal index may be obtained from the joint distribution of a finite number of observations, calculated at T distinct sets of variables.

c) The results obtained for the integer-valued max-autoregressive and moving average models generalize the ones obtained for the stationary case: whereas for the max-autoregressive model the extremal index is less than unity (reflecting the influence of the dependence structure on the extremes), for the moving averages the extremal index is equal to one.

2. PRELIMINARY RESULTS

The study of the extremal properties of stationary sequences is frequently based on the verification of appropriate dependence conditions which assure that the limiting distribution of the maximum term is of the same type as the limiting distribution of the maximum of i.i.d. rv's with the same marginal distribution F. For stationary sequences, usual conditions used in the literature are Leadbetter's $D(u_n)$ condition (Leadbetter *et al.* [23]) and condition $D^{(k)}(u_n)$, $k\in\mathbb{N}$, (Chernick *et al.* [13]). For completeness and reader's convenience the definition of condition $D(u_n)$ is given below.

Definition 2.1. The condition $D(u_n)$ is said to hold for a stationary sequence $(X_n)_{n\in\mathbb{N}}$ with marginal distribution F, if for any integers $i_1<\cdots<i_p<j_1<\cdots<j_q<n$ such that $j_1-i_p\geq l_n$ we have

$$\Big|F_{i_1,...,i_p,j_1,...,j_q}(u_n,...,u_n)-F_{i_1,...,i_p}(u_n,...,u_n)\,F_{j_1,...,j_q}(u_n,...,u_n)\Big|\;\leq\;\alpha_{n,l_n}$$

with $\alpha_{n,l_n}\underset{n\to\infty}{\longrightarrow}0$ for some sequence (l_n), $l_n=o(n)$.

For periodic sequences the following adaptation of condition $D^{(k)}(u_n)$ may be used:

Definition 2.2 (Ferreira and Martins [15])**.** Let $k\geq1$ be a fixed integer and $\mathbf{X}=(X_n)_{n\in\mathbb{N}}$ a T-periodic sequence verifying $D(u_n)$ with mixing coefficient α_{n,l_n}. The condition $D_T^{(k)}(u_n)$ holds for $\mathbf{X}$ if there exists a sequence of integers $(k_n)_{n\in\mathbb{N}}$ such that

$$\lim_{n\to\infty}k_n=+\infty\,,\qquad\lim_{n\to\infty}k_n\frac{l_n}{n}=0\,,\qquad\lim_{n\to\infty}k_n\,\alpha_{n,l_n}=0\;,$$

$$\lim_{n\to\infty}S^{(k)}_{[\frac{n}{k_nT}]}=0\;,$$

where

$$S^{(1)}_{[\frac{n}{k_nT}]}\;=\;\frac{n}{T}\sum_{i=1}^{T}\sum_{j=i+k}^{[\frac{n}{k_nT}]T}P\big(X_i>u_n,\,X_j>u_n\big)\,,$$

and for $k\geq2$

$$S^{(k)}_{[\frac{n}{k_nT}]}\;=\;\frac{n}{T}\sum_{i=1}^{T}\sum_{j=i+k}^{[\frac{n}{k_nT}]T}P\big(X_i>u_n,\,X_{j-1}\leq u_n<X_j\big)\,.$$

Remark 2.1. If $\lim_{n\to\infty} S^{(k)}_{[\frac{n}{k_n T}]} = 0$, then

$$\lim_{n\to\infty} \frac{n}{T} \sum_{i=1}^{T} P\big(X_i > u_n \geq M_{i+1,i+k-1},\, M_{i+k,[\frac{n}{k_n T}]T} > u_n\big) \;=\; 0\;,$$

with $M_{i,j} = \max_{i\leq r\leq j}(X_r)$ and $M_{i,j} = -\infty$ if $i > j$.

When $D(u_n)$ and $D^{(k)}_T(u_n)$ hold for a particular sequence the limiting distribution of the maximum term and its corresponding extremal index may be derived. Following Ferreira and Martins [15] the extremal index is given by

$$\theta \;=\; \lim_{n\to\infty} \frac{n\,\frac{1}{T}\sum\limits_{i=1}^{T} P\big(X_i > u_n \geq M_{i+1,i+k-1}\big)}{n\,\frac{1}{T}\sum\limits_{i=1}^{T} P\big(X_i > u_n\big)}\;.$$

Integer-valued sequences require extra care when the analysis of the extremal properties is in demand since in many cases, there is no non-degenerate limiting distribution for the maximum term. Anderson [6] defined a particular class of discrete distributions for which the maximum term (under an i.i.d. setting) possesses an almost stable behavior in the sense of the following theorem:

Theorem 2.1 (Anderson [6]). *Let F be a distribution function whose support consists of all sufficiently large integers. Then, there exists a sequence of constants (b_n) such that*

$$\begin{cases} \limsup\limits_{n\to\infty} F^n(x+b_n) \;\leq\; e^{-e^{-\alpha x}} \\ \liminf\limits_{n\to\infty} F^n(x+b_n) \;\geq\; e^{-e^{-\alpha(x-1)}} \end{cases},$$

for some $\alpha > 0$ and for every $x \in \mathbb{R}$, if and only if

$$\lim_{n\to\infty} \frac{1-F(n)}{1-F(n-1)} \;=\; \exp\{-\alpha\}\;.$$

In fact b_n may be obtained by $b_n = F_c^{-1}(1-\frac{1}{n})$ where F_c is any continuous distribution in the domain of attraction of the Gumbel distribution with $F_c([x]) = F_x$.

Whenever a distribution F satisfies the conditions of the theorem above we shall denote it by $F \in D_\alpha(Anderson)$. The study of stationary sequences with marginal distribution in the class of Anderson [6] was considered by Hall [17], who obtained the following result:

Theorem 2.2 (Hall [17]). *Suppose that for some $k \geq 1$, conditions $D(u_n)$ and $D^{(k)}(u_n)$ hold for the stationary sequence $\boldsymbol{X}$ with marginal $F \in D_\alpha$(Anderson), where $\boldsymbol{u_n}$ is a sequence of the form $u_n = x + b_n$. If $M_n = \max_{1\leq k\leq n}(X_k)$, then there exists a value $0 \leq \theta \leq 1$ such that*

$$\begin{cases} \limsup\limits_{n\to\infty} P\big(M_n \leq x + b_n\big) \leq e^{-\theta e^{-\alpha x}} \\ \liminf\limits_{n\to\infty} P\big(M_n \leq x + b_n\big) \geq e^{-\theta e^{-\alpha(x-1)}} \end{cases},$$

if and only if

$$P\big(M_{2,k} \leq u_n | X_1 > u_n\big) \underset{n\to\infty}{\longrightarrow} \theta \ .$$

Hall refers to the parameter θ as the extremal index due to its similarity with the conventional extremal index.

3. LIMITING DISTRIBUTION FOR THE MAXIMUM TERM

In this section attention is focused in the extremal behavior of periodic sequences with marginal distributions in Anderson's class. The first result extends Theorem 3 in Hall [17] for T-periodic integer-valued sequences.

Theorem 3.1. *Suppose that for $k \geq 1$ the conditions $D(u_n)$ and $D^k_T(u_n)$ hold for the T-periodic integer-valued sequence $\mathbf{X}$, with $F_r \in D_{\alpha_r}$(Anderson), for $r = 1, ..., T$ where $(u_n)_{n\in\mathbb{N}}$ is a sequence of the form $u_n = x + b_n$. If there exists $\underline{\theta}$ and $\overline{\theta}$, $0 \leq \underline{\theta} \leq \overline{\theta} \leq 1$, such that*

$$\begin{aligned} \underline{\theta} \;&=\; \liminf_{n\to\infty} \frac{\frac{n}{T} \sum\limits_{r=1}^{T} P\big(X_r > u_n > M_{r+1,r+k-1}\big)}{\frac{n}{T} \sum\limits_{r=1}^{T} P\big(X_r > u_n\big)} \\ &\leq\; \limsup_{n\to\infty} \frac{\frac{n}{T} \sum\limits_{r=1}^{T} P\big(X_r > u_n > M_{r+1,r+k-1}\big)}{\frac{n}{T} \sum\limits_{r=1}^{T} P\big(X_r > u_n\big)} \;=\; \overline{\theta} \ , \end{aligned}$$

then

$$\begin{cases} \limsup\limits_{n\to\infty} P\big(M_n \leq x + b_n\big) \leq e^{-\underline{\theta} \frac{1}{T} \sum_{r=1}^{T} e^{-\alpha_r x}} \\ \liminf\limits_{n\to\infty} P\big(M_n \leq x + b_n\big) \geq e^{-\overline{\theta} \frac{1}{T} \sum_{r=1}^{T} e^{-\alpha_r (x-1)}} \end{cases}.$$

Proof: First let us suppose that $\liminf_{n\to\infty} P(M_n \le x+b_n) > 0$, $\forall x$. By Proposition 2.1 in Ferreira and Martins [15] we have that

$$P\big(M_n \le u_n\big) - e^{-\frac{n}{T}\sum_{r=1}^{T} P(X_r > u_n > M_{r+1,r+k-1})} \to 0 , \qquad n\to\infty ,$$

which is equivalent to

$$P\big(M_n \le u_n\big) - \Big(e^{-\frac{n}{T}\sum_{r=1}^{T} P(X_r>u_n)}\Big)^{\frac{\frac{n}{T}\sum_{r=1}^{T} P(X_r>u_n>M_{r+1,r+k-1})}{\frac{n}{T}\sum_{r=1}^{T} P(X_r>u_n)}} \to 0 , \tag{3.1}$$

as $n\to\infty$. From Theorem 2.1 it follows that

$$\begin{aligned} 0 < e^{-\frac{1}{T}\sum_{r=1}^{T} e^{-\alpha_r(x-1)}} &\le \liminf_{n\to\infty} e^{-\frac{n}{T}\sum_{r=1}^{T} P(X_r>u_n)} \\ &\le \limsup_{n\to\infty} e^{-\frac{n}{T}\sum_{r=1}^{T} P(X_r>u_n)} \le e^{-\frac{1}{T}\sum_{r=1}^{T} e^{-\alpha_r x}} < 1 , \end{aligned}$$

and if we assume

$$\begin{aligned} \underline{\theta} &= \liminf_{n\to\infty} \frac{\frac{n}{T}\sum\limits_{r=1}^{T} P\big(X_r > u_n > M_{r+1,r+k-1}\big)}{\frac{n}{T}\sum\limits_{r=1}^{T} P(X_r > u_n)} \\ &\le \limsup_{n\to\infty} \frac{\frac{n}{T}\sum\limits_{r=1}^{T} P\big(X_r > u_n > M_{r+1,r+k-1}\big)}{\frac{n}{T}\sum\limits_{r=1}^{T} P\big(X_r > u_n\big)} = \overline{\theta} \end{aligned}$$

then, (3.1) leads to the stated result.

The case $P(M_n \le x+b_n) \to 0$ as $n\to\infty$ is easily handled by the results above and the arguments in Hall ([17], p. 725). We skip the details. □

As a consequence of Theorem 3.1 the extremal index can be computed as follows:

Corollary 3.1. *Suppose that for some* $k \ge 1$ *the conditions* $D(u_n)$ *and* $D_T^k(u_n)$ *hold for the T-periodic integer-valued sequence* $\mathbf{X}$, *with* $F_r \in D_{\alpha_r}$*(Anderson), for* $r=1,...,T$ *where* $\{u_n\}_{n\in\mathbb{N}}$ *is a sequence of the form* $u_n = x+b_n$. *Then, there exists a value* $0 \le \theta \le 1$ *such that*

$$\begin{cases} \limsup\limits_{n\to\infty} P\big(M_n \le x+b_n\big) \le e^{-\theta\frac{1}{T}\sum_{r=1}^{T} e^{-\alpha_r x}} \\ \liminf\limits_{n\to\infty} P\big(M_n \le x+b_n\big) \ge e^{-\theta\frac{1}{T}\sum_{r=1}^{T} e^{-\alpha_r (x-1)}} \end{cases} ,$$

if and only if

$$\frac{\frac{n}{T}\sum\limits_{r=1}^{T} P\big(X_r > u_n > M_{r+1,r+k-1}\big)}{\frac{n}{T}\sum\limits_{r=1}^{T} P\big(X_r > u_n\big)} \to \theta , \qquad n\to\infty .$$

4. MAX-AUTOREGRESSIVE PERIODIC SEQUENCES

Let $\mathbf{X}=(X_n)_{n\in\mathbb{N}}$ be a T-periodic non-negative integer-valued max-autoregressive sequence defined as

$$X_n \;=\; \max\{X_{n-1}, Z_n\} - c_n \;, \tag{4.1}$$

where $(c_1, ..., c_T) \in \mathbb{N}^T$, $c_{n+T} = c_n$ for all $n \in \mathbb{N}$ and $\mathbf{Z} = (Z_n)_{n\in\mathbb{N}}$ is a sequence of i.i.d. integer-valued rv's with common distribution F. Let H_n denote the distribution of X_n. The max-autoregressive sequence defined in (4.1) is an extension of the max-autoregressive model considered by Alpuim [3]. Her ideas will be extensively used throughout this section. First note that the following relations hold

$$\begin{aligned} H_n(x) \;&=\; P\big(X_n \le x\big) \;=\; P\big(X_{n-1} \le x + c_n,\, Z_n \le x + c_n\big) \\ &= \prod_{i=0}^{\infty} F\Big(x + \sum_{l=0}^{i} c_{n-l}\Big) \;=\; \prod_{s=0}^{T-1}\prod_{j=0}^{\infty} F\big(x + jS + S_{s,n}\big)\,, \end{aligned}$$

with $S = \sum_{i=1}^{T} c_i$ and $S_{s,n} = \sum_{l=0}^{s} c_{n-l}$. Moreover, it is also true that

$$F(x) = \frac{H_n(x - c_n)}{H_{n-1}(x)}\,, \qquad \text{for all } \; n \;. \tag{4.2}$$

Next result shows that if F belongs to Anderson's class then H_n will also belong to Anderson's class for all n.

Lemma 4.1. *Let* $\mathbf{X}$ *be a max-autoregressive integer-valued T-periodic sequence defined by* (4.1). *If* $F \in D_\alpha$*(Anderson) then* $H_n \in D_\alpha$*(Anderson),* $\forall\, n \in \mathbb{N}$. *Let* $u_n = x + b_n$ *be such that*

$$\begin{cases} \limsup\limits_{n\to\infty} n\big(1 - F(u_n)\big) \le e^{-\alpha x} \\ \liminf\limits_{n\to\infty} n\big(1 - F(u_n)\big) \ge e^{-\alpha(x-1)} \end{cases} .$$

Choosing $u'_n = x + b_n + \frac{\ln C_1}{\alpha}$ *with* $C_1 = \frac{\sum_{s=0}^{T-1} e^{-S_{s,1}\alpha}}{1-e^{-S\alpha}}$ *it follows that*

$$\begin{cases} \limsup\limits_{n\to\infty} n\big(1 - H_1(u'_n)\big) \le e^{-\alpha x} \\ \liminf\limits_{n\to\infty} n\big(1 - H_1(u'_n)\big) \ge e^{-\alpha(x-1)} \end{cases}$$

and

$$\begin{cases} \limsup\limits_{n\to\infty} n\big(1 - H_r(u'_n)\big) \le \gamma_{r,1}\, e^{-\alpha x} \\ \liminf\limits_{n\to\infty} n\big(1 - H_r(u'_n)\big) \ge \gamma_{r,1}\, e^{-\alpha(x-1)} \end{cases} ,$$

where

$$\gamma_{i,r} = \lim_{x\to\infty} \frac{1 - H_i(x)}{1 - H_r(x)}\,, \qquad r = 1, ..., T, \quad i = 0, ..., T-1 \;. \tag{4.3}$$

Furthermore for $i = 0, ..., T-1$

$$\begin{cases} \limsup_{n\to\infty} \dfrac{n}{T} \displaystyle\sum_{i=1}^{T} P\big(X_i > u'_n\big) \le \dfrac{1}{T} \displaystyle\sum_{i=1}^{T} \gamma_{i,1}\, e^{-\alpha x} \\ \liminf_{n\to\infty} \dfrac{n}{T} \displaystyle\sum_{i=1}^{T} P\big(X_i > u'_n\big) \ge \dfrac{1}{T} \displaystyle\sum_{i=1}^{T} \gamma_{i,1}\, e^{-\alpha (x-1)} \end{cases} .$$

Proof: First note that for any two integer-valued distribution functions, say F_1 and F_2, the following relation hold: If $F_1 \in D_\alpha(Anderson)$ and $\lim_{n\to\infty} \frac{1-F_2(n)}{1-F_1(n)} = c > 0$ then $F_2 \in D_\alpha(Anderson)$. Furthermore, if b_n is such that

$$\begin{cases} \limsup_{n\to\infty} n\big(1 - F_1(x+b_n)\big) \le e^{-\alpha x} \\ \liminf_{n\to\infty} n\big(1 - F_1(x+b_n)\big) \ge e^{-\alpha (x-1)} \end{cases} ,$$

then for $b'_n = b_n + \frac{\ln c}{\alpha}$

$$\begin{cases} \limsup_{n\to\infty} n\big(1 - F_2(x+b'_n)\big) \le e^{-\alpha x} \\ \liminf_{n\to\infty} n\big(1 - F_2(x+b'_n)\big) \ge e^{-\alpha (x-1)} \end{cases} .$$

Now suppose that $F \in D_\alpha(Anderson)$.

$$\begin{aligned} \lim_{x\to\infty} \frac{1 - H_n(x)}{1 - F(x)} &= \lim_{x\to\infty} \frac{\prod_{s=0}^{T-1} \prod_{j=0}^{\infty} F\big(x + jS + S_{s,n}\big)}{1 - F(x)} \\ &= \lim_{x\to\infty} \frac{\sum_{s=0}^{T-1} \sum_{j=0}^{\infty} 1 - F\big(x + jS + S_{s,n}\big)}{1 - F(x)} \\ &= \sum_{s=0}^{T-1} e^{-S_{s,n}\alpha} \lim_{x\to\infty} \sum_{j=0}^{\infty} \frac{1 - F(x+jS)}{1 - F(x)} . \end{aligned}$$

Since $\lim_{x\to\infty} \frac{1-F(x)}{1-F(x-1)} = e^{-\alpha}$ we may choose $\alpha' < \alpha$ so that there exists x_0 such that for all $x > x_0$ then $\frac{1-F(x+jS)}{1-F(x)} < e^{-jS\alpha'}$ for all j. By the dominated convergence theorem, limit and sum can be interchanged providing

$$\lim_{x\to\infty} \frac{1 - H_n(x)}{1 - F(x)} = \sum_{s=0}^{T-1} e^{-S_{s,n}\alpha} \sum_{j=0}^{\infty} e^{-jS\alpha} = \frac{\sum_{s=0}^{T-1} e^{-S_{s,n}\alpha}}{1 - e^{-S\alpha}} \equiv C_n .$$

Applying the relations stated in the beginning of the proof we conclude that $H_n \in D_\alpha(Anderson)$. □

We shall now obtain the asymptotic behaviour of the maximum term of the T-periodic non-negative integer-valued max-autoregressive sequence in (4.1).

Theorem 4.1. *Let* $\mathbf{X}$ *be the* T*-periodic non-negative integer-valued moving average sequence defined in* (4.1) *with* $F\in D_\alpha(Anderson)$*. If* $M_n=\max_{1\le k\le n}(X_k)$ *and* $u_n=x+b_n$ *with*

$$b_n = b_n' + \frac{\ln\left(\frac{1}{T}\sum_{i=1}^{T} C_i\right)}{\alpha}$$

where $C_i=\frac{\sum_{s=0}^{T-1} e^{-S_{s,i}\alpha}}{1-e^{-S\alpha}}$ *and* b_n' *is the sequence of normalizing constants of* F*, then*

$$\begin{cases} \limsup_{n\to\infty} P\big(M_n\le u_n\big) \le e^{-\theta e^{-\alpha x}} \\ \liminf_{n\to\infty} P\big(M_n\le u_n\big) \ge e^{-\theta e^{-\alpha(x-1)}} \end{cases}$$

and the extremal index θ *is given by*

$$\theta = \frac{\sum_{i=1}^{T} \gamma_{i,1}\big(1-\exp\{-\alpha\, c_{i+1}\}\big)}{\sum_{i=1}^{T}\gamma_{i,1}}\,, \tag{4.4}$$

with $\gamma_{i,1}=C_i/C_1$.

Proof: First we prove that condition $D(u_n)$ holds for $\mathbf{X}$. Note that for any two indexes i_1, i_2 we obtain the following relations by (4.2):

$$\begin{aligned} P\big(X_{i_1}\le x,\, X_{i_2}\le x\big) &= P\big(X_{i_1}\le x\big)\prod_{l=0}^{i_2-i_1-1} F(x+S_{l,i_2}) \\ &= H_{i_1}(x)\,\frac{H_{i_2}(x)}{H_{i_1}(x+S_{i_2-i_1-1,i_2})}\,. \end{aligned} \tag{4.5}$$

Using (4.5) we obtain

$$\begin{aligned} &\Big|H_{i_1,\dots,i_p,j_1,\dots,j_q}(u_n,\dots,u_n) - H_{i_1,\dots,i_p}(u_n,\dots,u_n)\,H_{j_1,\dots,j_q}(u_n,\dots,u_n)\Big| = \\ &\quad = \Bigg|H_{i_1}(u_n)\prod_{m=2}^{p}\prod_{l=0}^{i_m-i_{m-1}-1}F(u_n+S_{l,i_m})\prod_{m=2}^{q}\prod_{l=0}^{j_m-j_{m-1}-1}F(u_n+S_{l,j_m}) \\ &\qquad \times\Bigg(\prod_{l=0}^{l_n-1}F(u_n+S_{l,j_1})-H_{j_1}(u_n)\Bigg)\Bigg| \\ &\quad \le \left|\frac{H_{j_1}(u_n)}{H_{i_p}(u_n+S_{j_1-i_p-1})}-H_{j_1}(u_n)\right| \\ &\quad \le 1-H_{i_p}(u_n+S_{j_1-i_p-1}) \le 1-H_{i_p}(u_n)\,. \end{aligned}$$

Since $1-H_i(u_n)\sim O(\frac{1}{n})$ for all i, the desired result is obtained.

Next we show that condition $D''_T(u_n)$ also holds for $\mathbf{X}$.

$$
\begin{aligned}
P\big(X_i > u_n \ge X_{i+1},\, X_{i+j} > u_n\big) \;=&\\
&= P\big(X_i > u_n,\, X_{i+j} > u_n | X_{i+1} \le u_n\big)\, H_{i+1}(u_n)\\
&= P\big(X_i > u_n | X_{i+1} \le u_n\big)\, P\big(X_{i+j} > u_n | X_{i+1} \le u_n\big)\, H_{i+1}(u_n) \;,
\end{aligned}
$$

since the events $\{X_i > u_n | X_{i+1} \le u_n\}$ and $\{X_{i+j} > u_n | X_{i+1} \le u_n\}$ are independent for this type of sequences. Moreover

$$
\begin{aligned}
P\big(X_i > u_n | X_{i+1} \le u_n\big) \;&=\; \frac{H_{i+1}(u_n) - H_i(u_n)\, F(u_n + c_{i+1})}{H_{i+1}(u_n)}\\
&=\; 1 - \frac{H_i(u_n)}{H_{i+1}(u_n)}\, F(u_n + c_{i+1}) \;.
\end{aligned}
$$

Since $\frac{H_i(u_n)}{H_{i+1}(u_n)} \ge H_{i+1}(u_n)$ we have

$$
P\big(X_i > u_n | X_{i+1} \le u_n\big) \;\le\; 1 - H_i(u_n) \;=\; O\Big(\frac{1}{n}\Big) \;.
$$

For the second term we have

$$
\begin{aligned}
P\big(X_{i+j} > u_n | X_{i+1} \le u_n\big) \;&=\; 1 - \frac{H_{i+j}(u_n)}{H_{i+1}\Big(u_n + \sum\limits_{m=0}^{j-2} c_{i+j-m}\Big)}\\
&\le\; 1 - H_{i+j}(u_n)\\
&=\; O\Big(\frac{1}{n}\Big) \;.
\end{aligned}
$$

Hence

$$
\begin{aligned}
\lim_{n\to\infty} \frac{n}{T} \sum_{i=1}^{T} \sum_{j=i+2}^{[\frac{n}{k_n T}]T} P\big(X_i > u_n \ge X_{i+1},\, X_{i+j} > u_n\big) \;&\le\; \lim_{n\to\infty} nT\Big[\frac{n}{k_n T}\Big]\, O\Big(\frac{1}{n}\Big)\, O\Big(\frac{1}{n}\Big)\\
&=\; 0 \;.
\end{aligned}
$$

Note that by Corollary 3.1

$$
\begin{aligned}
\theta \;&=\; \lim_{n\to\infty} \frac{\frac{n}{T} \sum\limits_{i=1}^{T} P\big(X_i > u_n \ge X_{i+1}\big)}{\frac{n}{T} \sum\limits_{i=1}^{T} P\big(X_i > u_n\big)}\\
&=\; \lim_{n\to\infty} \frac{\sum\limits_{i=1}^{T} P\big(X_i > u_n \ge X_{i+1}\big) \,/\, P\big(X_1 > u_n\big)}{\sum\limits_{i=1}^{T} P\big(X_i > u_n\big) \,/\, P\big(X_1 > u_n\big)} \;.
\end{aligned}
$$

Since

$$\lim_{n\to\infty} \sum_{i=1}^{T} P\big(X_i > u_n\big) \,/\, P\big(X_1 > u_n\big) \;=\; \sum_{i=1}^{T} \gamma_{i,1}$$

and

$$\begin{aligned}
\lim_{n\to\infty} \sum_{i=1}^{T} P\big(X_i > u_n \geq X_{i+1}\big) \,/\, P\big(X_1 > u_n\big) \;&= \\
&= \lim_{n\to\infty} \sum_{i=1}^{T} \Big(P\big(X_i \leq u_n\big) - P\big(X_i \leq u_n,\, X_{i+1} \leq u_n\big)\Big) \,/\, P\big(X_1 > u_n\big) \\
&= \lim_{n\to\infty} \sum_{i=1}^{T} \Big(H_{i+1}(u_n) - H_i(u_n)\, F(u_n + c_{i+i})\Big) \,/\, \big(1 - H_1(u_n)\big) \\
&= \lim_{n\to\infty} \sum_{i=1}^{T} \bigg(H_{i+1}(u_n) - H_i(u_n)\, \frac{H_{i+1}(u_n)}{H_i(u_n + c_{i+i})}\bigg) \,/\, \big(1 - H_1(u_n)\big) \\
&= \lim_{n\to\infty} \sum_{i=1}^{T} \frac{H_{i+1}(u_n)}{H_i(u_n + c_{i+1})} \Big(H_i(u_n + c_{i+1}) - H_i(u_n)\Big) \,/\, \big(1 - H_1(u_n)\big) \\
&= \lim_{n\to\infty} \sum_{i=1}^{T} \frac{1 - H_i(u_n)}{1 - H_1(u_n)} \bigg(1 - \frac{1 - H_i(u_n + c_{i+1})}{1 - H_i(u_n)}\bigg) \\
&= \sum_{i=1}^{T} \gamma_{i,1} \,\big(1 - \exp(-\alpha\, c_{i+1})\big) \;,
\end{aligned}$$

then

$$\theta \;=\; \frac{\sum_{i=1}^{T} \gamma_{i,1} \big(1 - \exp(-\alpha\, c_{i+1})\big)}{\sum_{i=1}^{T} \gamma_{i,1}} \;,$$

concluding the proof. □

5. MOVING AVERAGE MODELS WITH EXPONENTIAL TYPE-TAILS

Let $\mathbf{Z} = (Z_n)_{n\in\mathbb{Z}}$ be a sequence of T-periodic integer-valued random variables. Throughout this section we will assume that

$$1 - F_{Z_r}(x) \;\sim\; K_r\, x^{\xi_r} (1+\lambda_r)^{-x} \;, \qquad x\to\infty, \quad \xi\in\mathbb{R}, \quad K_r, \lambda_r > 0 \;, \tag{5.1}$$

for $r = 1, ..., T$. Furthermore, we assume that X_n admits the representation

$$X_n \;=\; \sum_{j=-\infty}^{\infty} \beta_j \circ Z_{n-j} \;, \qquad \beta_j \in [0,1] \;, \tag{5.2}$$

where the discrete operator $\circ$ denotes binomial thinning defined as $\beta \circ Z = \sum_{s=1}^{Z} U_s(\beta)$, where $(U_s(\beta))$ is a i.i.d. sequence of Bernoulli random variables verifying $P\big(U_s(\beta)=1\big) = \beta$. Moreover, the sequence of coefficients $(\beta_j)_{j\in\mathbb{Z}}$ will be taken to satisfy

$$\sum_{j=-\infty}^{\infty} \beta_j < \infty \ ,$$

in order to ensure the almost sure convergence of (5.2). All thinning operations involved in (5.2) are independent, for each n. Nevertheless, dependence is allowed to occur between the thinning operators $\beta_j \circ Z_n$ and $\beta_i \circ Z_n$, $j \neq i$ (which belong to X_{n+j} and X_{n+i} respectively).

Lemma 5.1. *Under the conditions set above, the sum*

$$\sum_{j=-\infty}^{\infty} \beta_j \circ Z_{n-j} \ ,$$

with

$$\beta_j = O\big(|j|^{-\delta}\big) \ , \tag{5.3}$$

as $j \to \pm\infty$, *for some* $\delta > 2$, *converges almost surely to* X_n.

Proof: Note that

$$E\left[\sum_{j=-\infty}^{\infty} \beta_j \circ Z_{n-j}\right] \;=\; \sum_{s=0}^{T-1} E\big[Z_{n-s}\big] \sum_{j=-\infty}^{\infty} \beta_{jT+s} \;<\; \infty \ .$$

Likewise,

$$\begin{aligned} \mathrm{Var}\left[\sum_{j=-\infty}^{\infty} \beta_j \circ Z_{n-j}\right] \;&= \\ &= \sum_{s=0}^{T-1} \Big(\mathrm{Var}\big[Z_{n-s}\big] - E\big[Z_{n-s}\big]\Big) \sum_{j=-\infty}^{\infty} \beta_{jT+s}^2 + E[Z_{n-s}] \sum_{j=-\infty}^{\infty} \beta_{jT+s} \\ &< \ \infty \ . \end{aligned}$$

Thus $\sum_{j=-\infty}^{\infty} \beta_j \circ Z_{n-j} \to X_n$ almost surely by the Corollary of page 112 in Tucker [33]. □

We now begin with a series of results designed to understand the tail behavior of $X_r^{(s)} = \sum_{j=-\infty}^{\infty} \beta_{jT+s} \circ Z_{r-jT-s}$ as well as sums of these variables. The first result we present is a simple modification of Theorem 8 in Hall [19] for the stationary case, but crucial for the characterization of the tail behavior of X_r.

Lemma 5.2. *Let* $\mathbf{Z}$ *be a* T*-periodic sequence verifying* (5.1). *For fixed values of* $s=0,...,T-1$ *and* $r=1,...,T$, *it holds that, as* $x\to\infty$

$$P\big(X_r^{(s)}>x\big) \;\sim\; \breve{K}_{r-s}\, x^{\breve{\xi}_{r-s}}(1+\breve{\lambda}_{r-s})^{-x} \;,$$

for $\xi_{r-s}\neq 1$, *with* $\breve{\lambda}_{r-s}=\frac{\lambda_{r-s}}{\beta^{(s)}}$, $\beta^{(s)}=\max\limits_{-\infty\le j\le\infty}\{\beta_{jT+s}\}$, $k_s=\#\big\{j\colon \frac{\beta_{jT+s}}{\beta^{(s)}}=1\big\}$,

$$\breve{\xi}_{r-s} \;=\; \begin{cases} k_s\,\xi_{r-s}+k_s-1 & \xi_{r-s}>-1 \\ \xi_{r-s} & \xi_{r-s}<-1 \end{cases},$$

$$K^*_{r-s} \;=\; \beta^{(s)}\,K_{r-s}\bigg(\frac{1+\lambda_{r-s}}{\lambda_{r-s}+\beta^{(s)}}\bigg)^{\xi_{r-s}+1},$$

$$\breve{K}_{r-s} \;=\; \begin{cases} \breve{\lambda}_{r-s}^{k_s-1}\,K^{*\,k_s}_{r-s}\,\dfrac{\big(\Gamma(\xi_{r-s}+1)\big)^{k_s}}{\Gamma\big(k_s(\xi_{r-s}+1)\big)}\;E\Big[(1+\breve{\lambda}_{r-s})^{\sum_{j'\notin\gamma_s}\beta_{j'}\circ Z_{r-s}}\Big] & \xi_{r-s}>-1 \\ k_s\,K^*_{r-s}\Big(E\big[(1+\breve{\lambda}_{r-s})\big]\Big)^{k_s-1}E\Big[(1+\breve{\lambda}_{r-s})^{\sum_{j'\notin\gamma_s}\beta_{j'}\circ Z_{r-s}}\Big] & \xi_{r-s}<-1 \end{cases},$$

with $j'=jT+s$ *and* $\gamma_s=\big\{i_1,...,i_{k_s}\colon\; \beta_{i_h}=\beta^{(s)},\; h=1,...,k_s\big\}$.

We shall now obtain the tail behavior of F_{X_r}. For simplicity in notation we define $i_1,...,i_T=0,1,...,T-1$, being $i_1\neq i_2\neq ...\neq i_T$.

Lemma 5.3. *For the process defined in* (5.2) *it holds that, for* $r=1,...,T$, *as* $x\to\infty$,

$$P\big(X_r>x\big) \;\sim\; A_r^*\,x^{\xi_r^*}(1+\lambda_r^*)^{-x}\,, \tag{5.4}$$

with $\lambda_r^*=\min(\breve{\lambda}_r,...,\breve{\lambda}_{r-T+1})$. *Moreover, the constant* A_r^* *can be calculated as follows:*

1. *if* $\breve{\lambda}_{r-i_1}=\cdots=\breve{\lambda}_{r-i_T}$ *and*

(**a**) $\breve{\xi}_{r-i_1}=\cdots=\breve{\xi}_{r-i_T}<-1$ *then* $\xi_r^*=\breve{\xi}_{r-i_1}$ *and*

$$A_r^* \;=\; \begin{cases} C_{2,r} & T=2 \\ C_{T,r} & T\ge 3 \end{cases},$$

with

$$C_{2,r} \;=\; \breve{K}_{r-i_1}E\Big[(1+\lambda_r^*)^{X_r^{(i_2)}}\Big]+\breve{K}_{r-i_2}\,E\Big[(1+\lambda_r^*)^{X_r^{(i_1)}}\Big]\,,$$

$$C_{T,r} \;=\; C_{T-1,r}\,E\Big[(1+\lambda_r^*)^{X_r^{(i_T)}}\Big]+\breve{K}_{r-i_T}\,E\Big[(1+\lambda_r^*)^{\sum_{s=1}^{T-1}X_r^{(i_s)}}\Big]\,;$$

(**b**) $\breve{\xi}_{r-i_1} > -1$, ..., $\breve{\xi}_{r-i_T} > -1$ *then* $\xi^*_r = \sum_{s=1}^{T} \breve{\xi}_{r-i_s} + T - 1$, *and*

$$A^*_r = \begin{cases} C^*_{2,r} & T = 2 \\ C^*_{T,r} & T \geq 3 \end{cases},$$

with

$$C^*_{2,r} = \lambda^*_r \breve{K}_{r-i_1} \breve{K}_{r-i_2} \frac{\Gamma\big(\breve{\xi}_{r-i_1} + 1\big)\,\Gamma\big(\breve{\xi}_{r-i_2} + 1\big)}{\Gamma\big(\breve{\xi}_{r-i_1} + \breve{\xi}_{r-i_2} + 2\big)},$$

$$C^*_{T,r} = C^*_{T-1,r}\, \lambda^*_r \breve{K}_{r-i_T} \frac{\Gamma\left(\sum\limits_{s=1}^{T-1} \breve{\xi}_{r-i_s} + T - 1\right) \Gamma\big(\breve{\xi}_{r-i_T} + 1\big)}{\Gamma\left(\sum\limits_{s=1}^{T} \breve{\xi}_{r-i_s} + T\right)};$$

2. *if* $\breve{\lambda}_{r-i_1} < \cdots < \breve{\lambda}_{r-i_T}$, *then* $\xi^*_r = \breve{\xi}_{r-i_1}$ *and*

$$A^*_r = \breve{K}_{r-i_1} \prod_{h=2}^{T} E\Big[(1+\lambda^*_r)^{X_r^{(i_h)}}\Big];$$

3. *if* $\breve{\lambda}_{r-i_1} < \cdots < \breve{\lambda}_{r-i_{l+1}} = \cdots = \breve{\lambda}_{r-i_{l+k}} < \breve{\lambda}_{r-i_{l+k+1}} < \cdots < \breve{\lambda}_{r-i_T}$, *then* $\xi^*_r = \breve{\xi}_{r-i_1}$ *and*

$$A^*_r = \breve{Q}^{(k)}_{r-i_1} \left(\prod_{h=1}^{T-k-l} E\Big[(1+\lambda^*_r)^{X_r^{(i_{l+k+h})}}\Big] \right)$$

with

$$\breve{Q}^{(k)}_{r-i_1} = \breve{K}_{r-i_1} \left(\prod_{h=2}^{l} E\Big[(1+\lambda^*_r)^{X_r^{(i_h)}}\Big] \right) E\Big[(1+\lambda^*_r)^{\sum_{h=1}^{k} X_r^{(i_{l+h})}}\Big]; \tag{5.5}$$

4. *if* $\breve{\lambda}_{r-i_1} < \cdots < \breve{\lambda}_{r-i_l} < \breve{\lambda}_{r-i_{l+1}} = \cdots = \breve{\lambda}_{r-i_T}$ *then* $\xi^*_r = \breve{\xi}_{r-i_1}$ *and*

$$A^*_r = \breve{Q}^{(T-l)}_{r-i_1},$$

with $\breve{Q}^{(\cdot)}_{r-i_1}$ *defined as in* (5.5)*;*

5. *if* $\breve{\lambda}_{r-i_1} = \cdots = \breve{\lambda}_{r-i_l} < \breve{\lambda}_{r-i_{l+1}} < \cdots < \breve{\lambda}_{r-i_T}$ *and*

(**a**) $\breve{\xi}_{r-i_1} = \cdots = \breve{\xi}_{r-i_l} < -1$ *then* $\xi^*_r = \breve{\xi}_{r-i_1}$

$$A^*_r = \begin{cases} C_{2,r} & T = 2 \\ C_{T,r} \prod\limits_{h=l+1}^{T} E\Big[(1+\lambda^*_r)^{X_r^{(i_h)}}\Big] & 3 \leq l < T \end{cases};$$

(**b**) $\breve{\xi}_{r-i_1} > -1$, ..., $\breve{\xi}_{r-i_l} > -1$ *then* $\xi^*_r = \sum_{s=1}^{l} \breve{\xi}_{r-i_s} + l - 1$

$$A^*_r = \begin{cases} C^*_{2,r} & T = 2 \\ C^*_{T,r} \prod\limits_{h=l+1}^{T} E\Big[(1+\lambda^*_r)^{X_r^{(i_h)}}\Big] & 3 \leq l < T \end{cases}.$$

Proof: The result follows by applying repeatedly Lemma 7 in Hall [19] which is the discrete version of Theorem 7.1 in Rootzén [31], after some tedious calculations. □

We are now in conditions to obtain the limiting distribution of the maximum term of $\mathbf{X}$. An explicit expression for the sequence of norming constants (b_n) can be obtained though the following result. For clarification in notation we define $\check{\lambda} = \min_{1\le r\le T}\{\lambda_r^*\}$ and $(q_1, ..., q_k)$ the set of indices such that $\check{\lambda}/\lambda_{q_l}^* = 1$, for $l = 1, ..., k$ $(k \le T)$. In addition, we define $\check{\xi} = \max_{1\le l\le k}\{\xi_{q_l}^*\}$ and the set of indices $(p_1, ..., p_s)$ such that $\check{\xi}/\xi_{p_l}^* = 1$, with $l = 1, ..., s$, $(s \le k)$. Furthermore, let $A = \frac{1}{T}\sum_{j=1}^{s} A_{p_j}^*$.

Lemma 5.4. *For the T-periodic integer-valued sequence $\mathbf{X}$ given in* (5.2) *the normalizing constants b_n of Theorem 3.1 are given by*

$$b_n = \big(\ln(1+\check{\lambda})\big)^{-1} (\ln n + \check{\xi} \ln\ln n + \ln A) . \tag{5.6}$$

The demonstration of this lemma is based on the following result.

Lemma 5.5. *If a distribution function F belongs to the domain of attraction of an extreme value distribution, $(F \in D(G_\gamma(x)))$ and $F_* = F(x)(1 + \epsilon(x))$ with $\lim_{x\to x_F} \epsilon(x) = 0$, then $F_* \in D(G_\gamma(x))$.*

Proof of Lemma 5.4: By Lemma 5.3, as $x \to \infty$

$$\begin{aligned}\frac{1}{T}\sum_{r=1}^{T} P\big(X_r > x\big) &\sim \frac{1}{T}\sum_{r=1}^{T} A_r^* \, x^{\xi_r^*} (1+\lambda_r^*)^{-x} \\ &= A\, x^{\check{\xi}} (1+\check{\lambda})^{-x} \left[1 + \sum_{l=s+1}^{T} \frac{A_{p_j}^*}{A} \left(\frac{1+\lambda_{q_l}^*}{1+\check{\lambda}}\right)^{-x} x^{\xi_{q_l}^* - \check{\xi}}\right] \\ &\sim A\, x^{\check{\xi}} (1+\check{\lambda})^{-x} ,\end{aligned}$$

where the last step is justified by Lemma 5.5. □

Let $\hat{\mathbf{X}}$ be the associated independent T-periodic sequence of $\mathbf{X}$, i.e. $\hat{X}_1$, $\hat{X}_2$, ..., are independent random variables being the tail distribution of $\hat{X}_r$ as in (5.4) for $r = 1, ..., T$, and define $\hat{M}_n = \max(\hat{X}_n)$. Next result ensures that condition $D(u_n)$ holds for $\mathbf{X}$ with F_{Z_r} given as in (5.1).

Lemma 5.6. *Suppose that the T-periodic integer-valued sequence $\mathbf{X}$ given in* (5.2) *is defined by a.s. convergent sums and satisfies*

$$\begin{cases} \limsup\limits_{n\to\infty} P\big(\hat{M}_n \le x + b_n\big) \le e^{-\frac{1}{T}\sum_{r=1}^{T}(1+\lambda_r)^{-x}} \\ \liminf\limits_{n\to\infty} P\big(\hat{M}_n \le x + b_n\big) \ge e^{-\frac{1}{T}\sum_{r=1}^{T}(1+\lambda_r)^{-(x-1)}} \end{cases},$$

for all $x \in \mathbb{R}$ and some set of constants $\lambda_1, ..., \lambda_T > 0$, $b_n \in \mathbb{R}$. Then condition $D(x + b_n)$ holds for $\mathbf{X}$.

Proof: For any $\epsilon_n > 0$,

$$\sup_{i,j} \Big| F_{i_1,\dots,i_p,j_1,\dots,j_q}(u_n,\dots,u_n) - F_{i_1,\dots,i_p}(u_n,\dots,u_n)\, F_{j_1,\dots,i_q}(u_n,\dots,u_n) \Big| \;\le$$

$$\le\; \frac{n}{T} \sum_{r=1}^{T} P\Big(x + b_n - 2\,\epsilon_n < X_r \le x + b_n + 2\,\epsilon_n\Big)$$

$$+\; \frac{n}{T} \sum_{r=1}^{T} P\left(\left|\sum_{s=0}^{T-1} \sum_{j=[n^{\gamma_T}]+1}^{\infty} \beta_{jT+s} \circ Z_{r-jT-s}\right| > \epsilon_n\right)$$

$$+\; \frac{n}{T} \sum_{r=1}^{T} P\left(\left|\sum_{s=0}^{T-1} \sum_{j=-\infty}^{-[n^{\gamma_T}]-1} \beta_{jT+s} \circ Z_{r-jT-s}\right| > \epsilon_n\right)$$

where $j_1 - i_p \ge 2\, n^{\gamma_T}$, $\gamma \in (0,1)$. Note that

$$\frac{n}{T} \sum_{r=1}^{T} P\Big(x + b_n - 2\,\epsilon_n < X_r \le x + b_n + 2\,\epsilon_n\Big) \;=$$

$$=\; \frac{n}{T} \sum_{r=1}^{T} P\Big(X_r > x + b_n - 2\,\epsilon_n\Big) \;-\; \frac{n}{T} \sum_{r=1}^{T} P\Big(X_r > x + b_n + 2\,\epsilon_n\Big)\,.$$

Since $b_n \to \infty$ and $\epsilon \to 0$, if b_n is a normalizing constant for the maximum term, then $b_n^{\pm} = b_n \pm 2\,\epsilon_n$ are also constants for the maximum term. For each n and a fixed value of $r = 1,\dots,T$ $\;n\,P(X_r > x + b_n^-)$ and $n\,P(X_r > x + b_n^-)$ are step functions of x, with the same step width, and different location parameters, but whose difference converges to zero. Then

$$\frac{n}{T} \sum_{r=1}^{T} P\Big(x + b_n - 2\,\epsilon_n < X_r \le x + b_n + 2\,\epsilon_n\Big) \;\to\; 0\;, \qquad n \to \infty\;,$$

for all $x \in \mathbb{R}$, providing that

$$\frac{n}{T} \sum_{r=1}^{T} P\left(\left|\sum_{s=0}^{T-1} \sum_{j=[n^{\gamma_T}]+1}^{\infty} \beta_{jT+s} \circ Z_{r-jT-s}\right| > \epsilon_n\right) \;\to\; 0 \tag{5.7}$$

$$\frac{n}{T} \sum_{r=1}^{T} P\left(\left|\sum_{s=0}^{T-1} \sum_{j=-\infty}^{-[n^{\gamma_T}]-1} \beta_{jT+s} \circ Z_{r-jT-s}\right| > \epsilon_n\right) \;\to\; 0 \tag{5.8}$$

as $n \to \infty$, for some $\gamma_T \in (0,1)$ and $\epsilon = o(1)$ as $n \to \infty$, are sufficient conditions for $D(u_n)$. In proving (5.7) and (5.8) note that by Markov's inequality

$$\frac{n}{T} \sum_{r=1}^{T} P\left(\left|\sum_{s=0}^{T-1} \sum_{j=[n^{\gamma_T}]+1}^{\infty} \beta_{jT+s} \circ Z_{r-s}\right| > \epsilon_n\right) \;\le$$

$$\le\; \frac{n}{T} \sum_{r=1}^{T} \frac{E\left[\left(\sum_{s=0}^{T-1} \sum_{j=[n^{\gamma_T}]+1}^{\infty} \beta_{jT+s} \circ Z_{r-s}\right)^2\right]}{\epsilon_n^2}\;.$$

Using the properties of the thinning operation, for a fixed value $r=1,...,T$

$$
\begin{aligned}
E\left[\left(\sum_{s=0}^{T-1}\sum_{j=[n^{\gamma_T}]+1}^{\infty}\beta_{jT+s}\circ Z_{r-s}\right)^2\right] &= \\
&= \sum_{s=0}^{T-1}\Big(\mathrm{Var}\big[Z_{r-s}\big]-E\big[Z_{r-s}\big]\Big)\sum_{j=[n^{\gamma_T}]+1}^{\infty}\beta^2_{jT+s} \\
&\quad+\left(\sum_{s=0}^{T-1}E\big[Z_{r-s}\big]\sum_{j=[n^{\gamma_T}]+1}^{\infty}\beta_{jT+s}\right)^2+\sum_{s=0}^{T-1}E\big[Z_{r-s}\big]\sum_{j=[n^{\gamma_T}]+1}^{\infty}\beta_{jT+s} \\
&= O\Big(n^{-\gamma_T(\delta-1)}\Big)\ ,
\end{aligned}
$$

by (5.3). Hence by taking for instance $\epsilon_n = O\big((\ln n)^{-\zeta}\big)$, $\zeta>0$ and choosing $\gamma_T\in(0,1)$ such that $\gamma_T(\delta-1)>1$, we have that condition (5.7) is satisfied. For the expression in (5.8) the procedure is analogous. □

Next result provides sufficient conditions for $D'_T(u_n)$.

Lemma 5.7. *Denote* $n'_T=[n^{\gamma_T}]$ *and suppose that for some constants* $\gamma_T\in(0,1)$ *and* $\zeta>0$ *the following conditions hold, for* $u_n=x+b_n$, $\forall x\in\mathbb{R}$,

$$
\frac{n}{T}\sum_{r=1}^{T}\sum_{t=r+1}^{2n'_T}P\Big(X_r+X_t>2u_n\Big)\to 0\ ,\qquad n\to\infty\ ; \tag{5.9}
$$

$$
\frac{n^2}{T}\sum_{r=1}^{T}P\left(\sum_{s=0}^{T-1}\sum_{j=n'_T+1}^{\infty}\beta_{jT+s}\circ Z_{r-jT-s}>\zeta\right)\to 0\ ,\qquad n\to\infty\ ; \tag{5.10}
$$

$$
\frac{n^2}{T}\sum_{r=1}^{T}P\left(\sum_{s=0}^{T-1}\sum_{j=-\infty}^{-n'_T-1}\beta_{jT+s}\circ Z_{r-jT-s}>\zeta\right)\to 0\ ,\qquad n\to\infty\ ; \tag{5.11}
$$

$$
\sum_{s=0}^{T-1}\sum_{j=n'_T+1}^{\infty}\beta_{jT+s}\circ Z_{r-jT-s}\xrightarrow{P}0\,,\qquad \sum_{s=0}^{T-1}\sum_{j=-\infty}^{-n'_T-1}\beta_{jT+s}\circ Z_{r-jT-s}\xrightarrow{P}0\ . \tag{5.12}
$$

Then, condition $D'_T(u_n)$ *holds for the* T*-periodic integer-valued sequence* $\mathbf{X}$ *defined in* (5.2).

Proof: First note that, for a fixed value of $r=1,...,T$, $P(X_r>u_n, X_t>u_n) \le P(X_r+X_t>2\,u_n)$ following from (5.9) that

$$
\frac{n}{T}\sum_{r=1}^{T}\sum_{t=r+1}^{2n'_T}P\Big(X_r>u_n,\,X_t>u_n\Big)\to 0\ ,\qquad n\to\infty\ .
$$

Next write $X'_r = \sum_{s=0}^{T-1} \sum_{j=-\infty}^{n'_T} \beta_{jT+s} \circ Z_{r-jT-s}$ and $X''_t = \sum_{s=0}^{T-1} \sum_{j=n'_T}^{\infty} \beta_{jT+s} \circ Z_{t-jT-s}$ so that X'_r and X''_t are independent for $t > 2\,n'_T$. Following Rootzén [31], for a fixed value of $r = 1, ..., T$ it follows that

$$\begin{aligned} P\Big(X_r > u_n,\, X_t > u_n\Big) \;\le\; & P\Big(X'_r > u_n - \zeta\Big)\, P\Big(X''_r > u_n - \zeta\Big) \\ & + P\left(\sum_{s=0}^{T-1} \sum_{j=n'_T+1}^{\infty} \beta_{jT+s} \circ Z_{r-jT-s} \;>\; \zeta\right) \\ & + P\left(\sum_{s=0}^{T-1} \sum_{j=-\infty}^{-n'_T-1} \beta_{jT+s} \circ Z_{t-jT-s} \;>\; \zeta\right), \end{aligned}$$

and hence, writing $u^*_n = x - \zeta + b_n$ we have that

$$\begin{aligned} \frac{n}{T} \sum_{r=1}^{T} \sum_{t=2n'_T+1}^{[n/kT]T} & P\Big(X_r > u_n,\, X_t > u_n\Big) \;\le \\ & \le\; \sum_{r=1}^{T} \frac{n^2}{k\,T}\; P\left(\sum_{s=0}^{T-1} \sum_{j=-\infty}^{n'_T} \beta_{jT+s} \circ Z_{r-jT-s} \;>\; u^*_n\right) \\ & \quad \times P\left(\sum_{s=0}^{T-1} \sum_{j=-n'_T}^{\infty} \beta_{jT+s} \circ Z_{r-jT-s} \;>\; u^*_n\right) \\ & \quad + \frac{n^2}{T} \sum_{r=1}^{T} P\left(\sum_{s=0}^{T-1} \sum_{j=n'_T+1}^{\infty} \beta_{jT+s} \circ Z_{r-jT-s} \;>\; \zeta\right) \\ & \quad + \frac{n^2}{T} \sum_{r=1}^{T} P\left(\sum_{s=0}^{T-1} \sum_{j=-\infty}^{-n'_T-1} \beta_{jT+s} \circ Z_{r-jT-s} \;>\; \zeta\right). \end{aligned}$$

The last two terms tend to zero by (5.10) and (5.11). By the same line of reasoning as in Rootzén ([31], p. 622) it is easy to check that

$$\limsup_{n\to\infty} \frac{n}{T} \sum_{r=1}^{T} \sum_{t=2n'_T+1}^{[n/kT]T} P\Big(X_r > u_n,\, X_t > u_n\Big) \;\le\; \frac{1}{k} \times (\text{constant}) \;\to\; 0\,, \qquad k \to \infty\,. \;\square$$

The final result is formalized through the following theorem.

Theorem 5.1. *For the T-periodic integer-valued sequence $\mathbf{X}$ defined in* (5.2)*, with $k_s = 1$, $s = 0, ..., T-1$ and $\xi_{r-s} \neq 1$ for $r = 1 ..., T$, it holds that*

$$\begin{cases} \limsup\limits_{n\to\infty} P\big(M_n \le x + b_n\big) \;\le\; e^{-\frac{1}{T}\sum_{r=1}^{T}(1+\lambda^*_r)^{-x}} \\[2ex] \liminf\limits_{n\to\infty} P\big(M_n \le x + b_n\big) \;\ge\; e^{-\frac{1}{T}\sum_{r=1}^{T}(1+\lambda^*_r)^{-(x-1)}} \end{cases},$$

with b_n defined as in (5.6).

Proof: First note that for $r = 1, ..., T$

$$X_r + X_t \; = \sum_{s=0}^{T-1} \sum_{j=-\infty}^{\infty} \Big(\beta_{jT+s} \circ Z_{r-jT-s} \; + \; \beta_{jT+s+t} \circ Z_{r-jT-s} \Big) \,.$$

For simplicity in notation we define $\lambda_{\min} = \min(\lambda_r, ..., \lambda_{r-T+1})$ and for $s = 0, ..., T-1$, $\beta_1^{(s)} = \max_j \{\beta_{jT+s} \colon\, j \notin \gamma_s\} < \beta^{(s)}$, $\beta_2^{(s)} = \max_t \{\max_j \{\beta_{jT+s} + \beta_{jT+s+t}\}\} < 2\,\beta^{(s)}$, and $\breve\beta_{\max} = \max_{0 \le s \le T-1}\{\breve\beta^{(s)}\}$ with $\breve\beta^{(s)} = \max\{\beta_1^{(s)},\, \beta_2^{(s)}/2\}$.

$$\begin{aligned} E\Big[(1+h)^{\beta_{jT+s}\circ Z_{r-s} + \beta_{jT+s+t}\circ Z_{r-s}}\Big] \; &= \; E\Big[E\big[(1+h)^{\beta_{jT+s}\circ Z_{r-s} + \beta_{jT+s+t}\circ Z_{r-s}} | Z_{r-s}\big]\Big] \\ &= \; \tilde P_{Z_{r-s}}\Big(\beta(jT+s,t)\,h^2 + (\beta_{jT+s} + \beta_{jT+s+t})\,h\Big) \end{aligned}$$

with $0 \le h < \lambda_{\min}$. Since, for $h \ge 0$ and $s = 0, ..., T-1$, $\beta(jT+s,t)\,h^2 + (\beta_{jT+s} + \beta_{jT+s+t})\,h \le \breve\beta_{\max} h^2 + 2\,\breve\beta_{\max} h$, the existence of $E\big[(1+h)^{\beta_{jT+s}\circ Z_{r-s} + \beta_{jT+s+t}\circ Z_{r-s}}\big]$ will be granted if it is possible to find an $h > 0$ such that $\breve\beta_{\max} h^2 + 2\,\breve\beta_{\max} h < \lambda_{\min}$.

$$\breve\beta_{\max} h^2 + 2\,\breve\beta_{\max} h - \lambda_{\min} = 0 \quad \Longleftrightarrow \quad h = -1 \pm \sqrt{1 + \frac{\lambda_{\min}}{\breve\beta_{\max}}}\;.$$

Let $h_1 < 0 < h_2$ be the two solutions of this equation.

$$\begin{aligned} E\Big[(1+h)^{X_r + X_t}\Big] \; &= \; E\Big[(1+h)^{\sum_{s=0}^{T-1} \sum_{j=-\infty}^{\infty} (\beta_{jT+s}\circ Z_{r-jT-s} + \beta_{jT+s+t}\circ Z_{r-jT-s})}\Big] \\ &= \prod_{s=0}^{T-1} \Bigg(\prod_{j=-\infty}^{[t/2]} \tilde P_{Z_{r-s}}\Big(\beta(jT+s,t)\,h^2 + (\beta_{jT+s} + \beta_{jT+s+t})\,h\Big) \\ &\qquad \times \prod_{j=[t/2]+1}^{\infty} \tilde P_{Z_{r-s}}\Big(\beta(jT+s,t)\,h^2 + (\beta_{jT+s} + \beta_{jT+s+t})\,h\Big)\Bigg)\,. \end{aligned} \tag{5.13}$$

Moreover, $\tilde P'_{Z_{r-s}}(\nu) = E\big[(1+\nu)^{Z_{r-s}}\big] < \infty$, if $0 < \nu < \lambda_{\min}$, and $\tilde P'_{Z_{r-s}}(\nu) \ge 1$ for $0 \le \nu \le \breve\beta_{\max} h^2 + 2\,\breve\beta_{\max} h$. By the mean value Theorem, $\tilde P_{Z_{r-s}}(\nu_1 + \nu_2) \le \tilde P_{Z_{r-s}}(\nu_1)\,(1 + C\nu_2)$, $\nu_1, \nu_2 > 0$, $\nu_1 + \nu_2 \le \breve\beta_{\max} h^2 + 2\,\breve\beta_{\max} h$, with

$$C \,=\, \sup\left\{ \frac{\tilde P'_{Z_{r-s}}(\nu + x)}{\tilde P_{Z_{r-s}}(\nu)} \colon\; s = 0, ..., T-1,\; 0 < \nu + x < \breve\beta_{\max} h^2 + 2\breve\beta_{\max} h,\; \nu > 0,\; x > 0 \right\} \\ < \infty\;.$$

On the basis of this result we have for $\nu_1 = h(\beta_{jT+s} + \beta_{jT+s+t})$ and $\nu_2 = \beta(jT+s,t)h^2$

$$\begin{aligned} &\prod_{j=-\infty}^{[t/2]} \tilde P_{Z_{r-s}}\Big(\beta(jT+s,t)\,h^2 + (\beta_{jT+s} + \beta_{jT+s+t})\,h\Big) \;\le \\ &\quad \le \prod_{j=-\infty}^{[t/2]} \tilde P_{Z_{r-s}}\Big((\beta_{jT+s} + \beta_{jT+s+t})\,h\Big) \prod_{j=-\infty}^{[t/2]} \Big(1 + C\,\beta_{jT+s}\,\beta(jT+s,t)\,h^2\Big) \\ &\quad \le \prod_{j=-\infty}^{[t/2]} \tilde P_{Z_{r-s}}\big(\beta_{jT+s}\,h\big) \prod_{j=-\infty}^{-[t/2]} \Big(1 + C\,\beta_{jT+s}\,h\Big) \prod_{j=-\infty}^{[t/2]} \Big(1 + C\,\beta(jT+s,t)\,h^2\Big)\;. \end{aligned}$$

Noticing that $\tilde{P}_{Z_{r-s}}(\beta_{jT+s}\, h) = 1 + \beta_{jT+s}\, h\, E[Z_{r-s}]\,(1+o(1))$ and using (5.3), we may conclude that the last expression is bounded, uniformly in t. Using a similar argument for the second product in (5.13) we are lead to conclude that $E\big[(1+h)^{X_r+X_t}\big] < \infty$ for $r=1,...T$. By Lemma 5.4, $u_n \sim \frac{\ln n}{\ln(1+\tilde{\lambda})}$ as $n\to\infty$. By Bernstein's inequality

$$\begin{aligned}
P\Big(X_r+X_t > 2\,u_n\Big) \;&\le\; E\Big[(1+h)^{X_r+X_t}\Big]\,(1+h)^{-2u_n}\\
&= O\Big((1+h)^{-2u_n}\Big)\\
&= O\Big(n^{\frac{\ln(1+2h+h^2)}{\ln(1+\tilde{\lambda})}}\Big)\\
&= o\Big(n^{-(1+\gamma_T)}\Big)\,,
\end{aligned}$$

where the last equality follows by the arguments given in Hall ([19], p. 373). Moreover, in proving (5.10) and (5.11), it suffices to show by Bernstein's inequality that

$$E\bigg[(1+h)^{\sum_{s=0}^{T-1}\sum_{j=Tn'_T+1}^{\infty}\beta_{jT+s}\circ Z_{r-jT-s}}\bigg]$$

and

$$E\bigg[(1+h)^{\sum_{s=0}^{T-1}\sum_{j=-\infty}^{-Tn'_T-1}\beta_{jT+s}\circ Z_{r-jT-s}}\bigg]\,,$$

are bounded as $n\to\infty$, for some $h = n^{\eta}-1$, $\eta>0$. We can choose ζ and η such that $2 < \zeta\,\eta < \zeta\,\gamma_T(\delta-1)$. By (5.3), we have that

$$\begin{aligned}
E\bigg[(1+h)^{\sum_{s=0}^{T-1}\sum_{j=n'_T+1}^{\infty}\beta_{jT+s}\circ Z_{r-jT-s}}\bigg] &= \prod_{s=0}^{T-1}\prod_{j=Tn'_T+1}^{\infty} E\Big[(1+h)^{\beta_{jT+s}\circ Z_{r-jT-s}}\Big]\\
&= \prod_{s=0}^{T-1}\prod_{j=n'_T+1}^{\infty} \tilde{P}_{Z_{r-s}}(\beta_{jT+s}\, h)\\
&= \prod_{s=0}^{T-1}\prod_{j=n'_T+1}^{\infty}\Big(1+\beta_{jT+s}\, h\, E\big[Z_{r-s}\big]\big(1+o(1)\big)\Big)\\
&< \infty\,,
\end{aligned}$$

as $n\to\infty$ providing

$$\begin{aligned}
&\frac{n^2}{T}\sum_{r=1}^{T} P\left(\sum_{s=0}^{T-1}\sum_{j=n'_T+1}^{\infty}\beta_{jT+s}\circ Z_{r-jT-s} > \zeta\right) \;\le\\
&\qquad\le\; \frac{n^2}{T}\, E\bigg[(1+h)^{\sum_{s=0}^{T-1}\sum_{j=n'_T+1}^{\infty}\beta_{jT+s}\circ Z_{r-jT-s}}\bigg]\, n^{-\zeta\eta} \;\to\; 0\,, \qquad n\to\infty\,.
\end{aligned}$$

A similar procedure can be carried out to prove (5.11). We skip the details.

Finally, the proof is completed upon showing (5.12). Note that

$$\begin{aligned} E\left[\sum_{s=0}^{T-1}\sum_{j=n'_T+1}^{\infty}\beta_{jT+s}\circ Z_{r-jT-s}\right] &= \sum_{s=0}^{T-1} E\big[Z_{r-s}\big]\sum_{j=n'_T+1}^{\infty}\beta_{jT+s} \\ &< \sum_{s=0}^{T-1} E\big[Z_{r-s}\big]\sum_{j=n'_T+1}^{\infty} O(j^{-\delta}) \\ &= O\Big(n^{\gamma_T(-\delta+1)}\Big) \;\rightarrow\; 0\;, \qquad n\to\infty\;. \end{aligned}$$

Moreover

$$\begin{aligned} &\operatorname{Var}\left[\sum_{s=0}^{T-1}\sum_{j=n'_T+1}^{\infty}\beta_{jT+s}\circ Z_{r-jT-s}\right] = \\ &= \sum_{s=0}^{T-1}\sum_{j=n'_T+1}^{\infty}\beta^2_{jT+s}\Big(\operatorname{Var}\big[Z_{r-s}\big]-E\big[Z_{r-s}\big]\Big) + \sum_{s=0}^{T-1}\sum_{j=n'_T+1}^{\infty}\beta_{jT+s}\,E\big[Z_{r-s}\big] \\ &< \sum_{s=0}^{T-1}\sum_{j=n'_T+1}^{\infty}\Big(O(j^{-2\delta})+O(j^{-\delta})\Big) \\ &= O\Big(n^{\gamma_T(-\delta+1)}\Big) \;\rightarrow\; 0\;, \qquad n\to\infty\;. \end{aligned}$$

Hence, (5.12) holds concluding the proof. □

REFERENCES

[1] Al-Osh, M. and Alzaid, A. (1987). First order integer-valued autoregressive INAR(1) process, *J. Time Series Anal.*, **8**, 261–275.

[2] Al-Osh, M. and Alzaid, A. (1988). Integer-valued moving average (INMA) process, *Statistical Papers*, **29**, 281–300.

[3] Alpuim, M.T. (1988). An extremal Markovian sequence, *J. Appl. Probab.*, **26**, 219–232.

[4] Aly, E.-E. and Bouzar, N. (1994). Explicit stationary distributions for some Galton-Watson processes with immigration, *Commun. Statist.-Stochastic Models*, **10**, 499–517.

[5] Aly, E.-E. and Bouzar, N. (2005). Stationary solutions for integer-valued autoregressive processes, *International Journal of Mathematics and Mathematical Science*, **1**, 1–18.

[6] Anderson, C.W. (1970). Extreme value theory for a class of discrete distributions with applications to some stochastic processes, *J. Appl. Probab.*, **7**, 99–113.

[7] Ahn, S.; Gyemin, L. and Jongwoo, J. (2000). Analysis of the M/D/1-type queue based on an integer-valued autoregressive process, *Operational Research Letters*, **27**, 235–241.

[8] Blundell, R.; Griffith, R. and Windmeijer, F. (2002). Individual effects and dynamics in count data models, *Journal of Econometrics*, **108**, 113–131.

[9] Brännäs, K. (1995). Explanatory variables in the AR(1) count data model, *Umeå Economic Studies*, **381**.

[10] Brännäs, K. and Hall, A. (2001). Estimation in integer-valued moving average models, *Appl. Stochastic Models Bus. Ind.*, **17**, 277–291.

[11] Brännäs, K. and Hellström, J. (2001). Generalized integer-valued autoregression, *Econometric Reviews*, **20**, 425–443.

[12] Brännäs, K.; Hellström, J. and Nordström, J. (2002). A new approach to modelling and forecasting monthly guest nights in hotels, *Int. J. Forecast.*, **18**, 9–30.

[13] Chernick, M.; Hsing, T. and McCormick, W.J. (1991). Calculating the extremal index for a class of stationary sequences, *Adv. Appl. Prob.*, **6**, 711–731.

[14] Du, J.-G. and Li, Y. (1991). The integer valued autoregressive (INAR(p)) model, *J. Time Series Anal.*, **12**, 129–142.

[15] Ferreira, H. and Martins, A.P. (2003). The extremal index of sub-sampled sequences with strong local dependence, *Revstat*, **1**, 16–24.

[16] Garcia-Ferrer, A. and Queralt, R.A. (1997). A note on forecasting international tourism deman in Spain, *Int. J. Forecast.*, **13**, 539–549.

[17] Hall, A. (1996). Maximum term of a particular autoregressive sequence with discrete margins, *Commun. Statist.-Theory Meth.*, **25**, 721–736.

[18] Hall, A. (2001). Extremes of integer-valued moving averages models with regularly varying tails, *Extremes*, **4**, 219–239.

[19] Hall, A. (2003). Extremes of integer-valued moving averages models with exponential type-tails, *Extremes*, **6**, 361–379.

[20] Hall, A. and Moreira, O. (2006). A note on the extremes of a particular moving average count data model, *Statist. Probab. Lett.*, **76**, 135–141.

[21] Hooghiemstra, G.; Meester, L.E. and Hüsler, J. (1998). On the extremal index for the $M/M/s$ queue, *Commun. Statist.-Stochastic Models*, **14**, 611–621.

[22] Latour, A. (1988). Existence and stochastic structure of a non-negative integer-valued autoregressive processes, *J. Time Ser. Anal.*, **4**, 439–455.

[23] Leadbetter, M.R.; Lindgren, G. and Rootzén, H. (1983). *Extremes and Related Properties of Random Sequences and Processes*, Springer-Verlag, New York.

[24] McCabe, B.P.M. and Martin, G.M. (2005). Bayesian Prediction of low count time series, *Int. J. Forecast.*, **21**, 315–330.

[25] McCormick, W.P. and Park, Y.S. (1992). Asymptotic analysis of extremes from autoregressive negative binomial processes, *J. Appl. Probab.*, **29**, 904–920.

[26] McKenzie, E. (1985). Some simple models for discrete variate time series, *Water Res. Bull.*, **21**, 645–650.

[27] McKenzie, E. (1986). Autoregressive analysis of extremes from autoregressive negative binomial processes, *J. Appl. Probab.*, **29**, 904–920.

[28] McKenzie, E. (2003). *Discrete variate time series*, In "Handbook of Statistics" (D.N. Shanbhag and C.R. Rao, Eds.), Elsevier Science, 573–606.

[29] Nordström, J. (1996). Tourism satellite account for Sweden 1992-93, *Tourism Economics*, **2**, 13–42.

[30] Quoreshi, A.M.M.S. (2006). Bivariate time series modelling of financial count data, *Commun. Statist.-Theory Meth.*, **35**, 1343–1358.

[31] Rootzén, H. (1986). Extreme value theory for moving average processes, *Ann. Probab.*, **14**, 612–652.

[32] Silva, I.; Silva, M.E.; Pereira, I. and Silva, N. (2005). Replicated INAR(1) processes, *Methodology and Computing in Applied Probability*, **7**, 517–542.

[33] Tucker, H. (1967). *A Graduate Course in Probability*, Academic Press, New York.

[34] Zhou, J. and Basawa, I.V. (2005). Least-squared estimation for bifurcation autoregressive processes, *Statist. Probab. Lett.*, **74**, 77–88.

[35] Zhu, R. and Joe, H. (2003). A new type of discrete self-decomposability and its applications to continuous-time Markov processes for modeling count data time series, *Stoch. Models*, **19**, 235–254.