REVSTAT – Statistical Journal
Volume 7, Number 2, June 2009, 189–201

A SADDLEPOINT APPROXIMATION TO A DISTRIBUTION-FREE TEST FOR STOCHASTIC ORDERING IN THE COMPETING RISKS MODEL

Author: Hidetoshi Murakami
– Department of Industrial and Systems Engineering, Chuo University,
1-13-27 Kasuga, Bunkyo-ku, Tokyo, 112-8551 Japan
murakami@indsys.chuo-u.ac.jp

Received: March 2009 Revised: May 2009 Accepted: May 2009

Abstract:

- The approximation for the distribution function of a test statistic is extremely important in statistics. A distribution-free test for stochastic ordering in the competing risks model has been proposed by Bagai *et al.* (1989). Herein, we performed a standard saddlepoint approximation in the tails for the Bagai statistic under finite sample sizes. We then compared the saddlepoint approximation with the Bagai approximation to obtain the exact critical value. The table of the critical values was extended by using the saddlepoint approximation. Additionally, the orders of errors of a saddlepoint approximation were derived.

Key-Words:

- *saddlepoint approximation; moment generating function; Bagai statistic; distribution-free test.*

AMS Subject Classification:

- 62F03, 62G10.

1. INTRODUCTION

Testing hypotheses is one of the most important challenges in performing nonparametric statistics. Various nonparametric statistics have been proposed and discussed over the course of many years. However, the single most challenging testing problem lies in calculating the exact critical test statistic value for small data sets. It is also difficult to obtain the exact critical value when the sample sizes are moderate to large in size. Under these circumstances, we must estimate the exact critical value with an approximation method. Hence, considering approximations for evaluating the density or distribution function of the test statistic remains one of the most important topics in statistics. For the approximation presented in this study, we used a saddlepoint formula proposed by Daniels (1954, 1987) and of the type developed by Lugannani and Rice (1980). The saddlepoint approximation can be obtained for any statistic that admits a cumulant generating function. Additionally, for small sample sizes, the saddlepoint can generate accurate probabilities in the distribution tails. Saddlepoint approximations have been used with great success by many authors, and excellent discussions of their applications to a range of distributional problems are provided by Reid (1988), Jensen (1995), Goutis and Casella (1999), Huzurbazar (1999), Kolassa (2006) and Butler (2007). Additionally, Easton and Ronchetti (1986) have discussed saddlepoint approximations by using expansions of the cumulant generating function. For conducting a distribution-free test, Giles (2001) and Chen and Giles (2008) compared saddlepoint approximations with the limiting distribution of the Anderson–Darling (1952, 1954) and the Sinclair and Spurr (1988) tests and found that the saddlepoint approximations were better than both. In nonparametric statistics, researchers are very interested in considering approximations under finite sample sizes. Froda and van Eeden (2000) proposed a uniform saddlepoint expansion to the null distribution of the Wilcoxon–Mann–Whitney test (Gibbons and Chakraborti, 2003). Additionally, Bean *et al.* (2004) compared a saddlepoint approximation of the Wilcoxon–Mann–Whitney test with that of Edgeworth, and determined normal and uniform approximations under finite sample sizes.

In addition to assessing distributions, nonparametric statistics are used to test the competing risks model. Various authors have proposed the test statistics for the case in which there are competing risks and without censoring. For example, Bagai *et al.* (1989) developed distribution-free rank tests for stochastic ordering in the two independent competing risks model. Yip and Lam (1992) suggested a class of weighted log-rank-type statistics, and Neuhaus (1991) constructed the asymptotically optimal rank tests for q competing risks against stochastic ordering without censoring. Hu and Tsai (1999) considered the linear rank tests for a competing risks model. In this paper, we considered a saddlepoint approximation to the small size sample distribution of the statistic proposed by Bagai *et al.* (1989), and we estimated the exact critical value of the Bagai statistic for large sample sizes, also using a saddlepoint approximation. We expect to apply the saddlepoint approximation to the other statistics for testing stochastic ordering in the competing risks model. In Section 2, we introduce Bagai's (1989) statistic, namely V, and the moment-generating function of the V statistic. In Section 3, we report on a saddlepoint approximation to the distribution of the V statistic and compare a saddlepoint approximation with that of Bagai. In addition, we extend the table of the critical values of the Bagai statistic using the saddlepoint approximation. In Section 4, we report on the derivation of the orders of the errors of a saddlepoint approximation.

2. THE BAGAI STATISTIC

In this section, we introduce a distribution-free test for stochastic ordering in a two independent competing risks model. We assumed that X and Y are independent and absolutely continuous random variables. Let $X_1, ..., X_n$ and $Y_1, ..., Y_n$ be two random samples of independent observations of size n, each of which has a continuous distribution described as F and G, respectively, denoting the hypothetical times to failure of the n individuals in the sample under the two risks. We observed only $(T_1, \delta_1), ..., (T_n, \delta_n)$, where $T_i = \min(X_i, Y_i)$ denotes the time to failure and $\delta_i = I(X_i > Y_i)$ indicates the cause of failure of the i-th unit. On the basis of these data types, we were interested in testing the hypothesis:

$$H_0\colon\ F(x) = G(x) \qquad \text{against} \qquad H_1\colon\ F(x) \leqslant G(x)\ .$$

Subsequently, Bagai *et al.* (1989) proposed a test statistic, namely V, as

$$V \;=\; 2\sum_{i=1}^{n} (2\,n-1-R_i)\,\delta_i \;-\; \frac{3\,n\,(n-1)}{2}\ .$$

Here, R_i denotes the rank of T_i among $T_1, ..., T_n$. In addition, the moment-generating function of the V statistic is given by Bagai *et al.* (1989) as follows:

$$M^*(s) \;=\; 2^{-n} \exp\!\left(\frac{-3\,n\,(n-1)\,s}{2}\right) \prod_{j=2}^{n+1} \Big\{1-\exp\!\big(2\,s\,(2\,n-j)\big)\Big\}\ .$$

However, there is a typo in the $M^*(s)$ formula given by Bagai *et al.* (1989), and we should use the moment-generating function $M(s)$ as follows:

$$M(s) \;=\; 2^{-n} \exp\!\left(\frac{-3\,n\,(n-1)\,s}{2}\right) \prod_{j=2}^{n+1} \Big\{1+\exp\!\big(2\,s\,(2\,n-j)\big)\Big\}\ .$$

By using the moment-generating function $M(s)$, the mean and variance of the V statistic are respectively given by

$$\mathrm{E}(V) = 0 \qquad \text{and} \qquad \mathrm{var}(V) = \frac{n\,(n-1)\,(14\,n-13)}{6}$$

under the null hypothesis. We consider a saddlepoint approximation by using the moment-generating function $M(s)$ in the next section.

3. SADDLEPOINT APPROXIMATION

3.1. Saddlepoint approximation to Bagai statistic

In this section, we considered a saddlepoint approximation (Daniels, 1954, 1987) to the distribution of the V statistic. In the previous section, the moment-generating function $M(s)$ was given by

$$M(s) \;=\; 2^{-n} \exp\left(\frac{-3\,n\,(n-1)\,s}{2}\right) \prod_{j=2}^{n+1} \Big\{1 + \exp\big(2\,s\,(2n-j)\big)\Big\}\,.$$

The cumulant generating function of the V statistic, namely $\kappa(s)$, is

$$\kappa(s) \;=\; \log\big[M(s)\big] \;=\; -n\log 2 - \frac{3\,n\,(n-1)\,s}{2} + \sum_{j=2}^{n+1} \log\Big\{1 + \exp\big(2\,s\,(2n-j)\big)\Big\}\,.$$

To obtain the saddlepoint approximation, we evaluated the first two derivatives of the cumulant generating function as

$$\kappa^{(1)}(s) \;=\; -\frac{3\,n\,(n-1)}{2} + \sum_{j=2}^{n+1} \frac{2\,(2n-j)\exp\big(2\,s\,(2n-j)\big)}{1+\exp\big(2\,s\,(2n-j)\big)}$$

and

$$\kappa^{(2)}(s) \;=\; \sum_{j=2}^{n+1} \frac{4\,(j-2n)^2\exp\big(2\,s\,(j+2n)\big)}{\big(\exp(2\,j\,s)+\exp(4n\,s)\big)^2}\;,$$

where $\kappa^{(i)}(\cdot)$ denotes the i-th derivative. A highly lucid account of the generalized Lugannani and Rice formula for nonnormal distributions was suggested by Wood *et al.* (1993). Then, to determine the saddlepoint approximation to $\Pr(V \geq v)$, we solved the saddlepoint equation, $\kappa^{(1)}(s) = v$, and used the unique solution $(s = \hat{s})$ to calculate

$$\hat{w} = \sqrt{2\left(\hat{s}\,v - \kappa(\hat{s})\right)}\;\mathrm{sgn}(\hat{s}) = \sqrt{2\left(\hat{s}\,\kappa^{(1)}(\hat{s}) - \kappa(\hat{s})\right)}\;\mathrm{sgn}(\hat{s}) \quad \text{and} \quad \hat{u} = \hat{s}\,\sqrt{\kappa^{(2)}(\hat{s})}\;,$$

given by Wood *et al.*, where $\mathrm{sgn}(\hat{s}) = \pm 1, 0$ if $\hat{s}$ is positive, negative, or zero. The saddlepoint approximation to the cumulative distribution function of the V statistic is

$$\Pr(V \geq v) \;\approx\; 1 - \Phi(\hat{w}) + \phi(\hat{w})\left\{\frac{1}{\hat{u}} - \frac{1}{\hat{w}}\right\},$$

where $\phi(\cdot)$ is the standard normal density function and $\Phi(\cdot)$ is the corresponding cumulative distribution function.

3.2. Numerical results

In this section, we report on the evaluation of the tail probability using the saddlepoint approximation. For this test, we listed the exact probability of the V statistic derived by Bagai *et al.*, namely E_P, the Bagai's approximation, namely A_B, and a saddlepoint approximation, namely A_S, given in Tables 1 and 2.

Table 1-1: Numerical results for 1% significance level.

n	v	E_P	A_B	A_S
7	51	0.0156	0.0183	0.016256
8	68	0.0117	0.0126	0.010587
9	84	0.0117	0.0113	0.009430
10	99	0.0107	0.0117	0.010029
11	115	0.0102	0.0118	0.010393
12	134	0.0105	0.0186	0.009538
13	152	0.0102	0.0109	0.009701
14	169	0.0106	0.0116	0.010537
15	191	0.0103	0.0107	0.009674
16	210	0.0105	0.0111	0.010161
17	232	0.0100	0.0181	0.009895
18	255	0.0100	0.0105	0.009595
19	275	0.0104	0.0110	0.010205
20	298	0.0104	0.0109	0.010201

Table 1-2: Difference between E_P and approximations.

n	$\|E_P - A_B\|$	$\|E_P - A_S\|$
7	0.0027	**0.000656**
8	**0.0009**	0.001113
9	**0.0004**	0.002270
10	0.0010	**0.000671**
11	0.0016	**0.000193**
12	0.0081	**0.000962**
13	0.0007	**0.000499**
14	0.0010	**0.000063**
15	**0.0004**	0.000626
16	0.0006	**0.000339**
17	0.0081	**0.000105**
18	0.0005	**0.000405**
19	0.0006	**0.000195**
20	0.0005	**0.000199**

The numeric results for 1% and 5% significance levels are listed in Tables 1-1 and 2-1, respectively. The difference between the exact probability of the V statistic and the approximations is given in Tables 1-2 and 2-2, respectively, where the best result is in bold. Note that v and n denote the exact critical value of the Bagai statistic and the sample size, respectively. We treated the cases $7 \leq n \leq 20$ at a 1% significance level and $5 \leq n \leq 20$ at 5% significance, which represent the same cases as presented in Bagai *et al.* (1989).

Table 2-1: Numerical results for 5% significance level.

n	v	E_P	A_B	A_S
5	22	0.0625	0.0522	0.057870
6	31	0.0625	0.0499	0.051279
7	41	0.0547	0.0463	0.047075
8	50	0.0508	0.0499	0.050831
9	62	0.0508	0.0461	0.046546
10	73	0.0508	0.0472	0.047642
11	83	0.0527	0.0513	0.051902
12	98	0.0500	0.0467	0.046961
13	108	0.0528	0.0516	0.052143
14	123	0.0511	0.0494	0.049753
15	137	0.0516	0.0495	0.049831
16	150	0.0523	0.0512	0.051650
17	166	0.0519	0.0500	0.050448
18	181	0.0516	0.0506	0.050878
19	197	0.0516	0.0505	0.050748
20	214	0.0511	0.0499	0.050178

Table 2-2: Difference between E_P and approximations.

n	$\lvert E_P - A_B \rvert$	$\lvert E_P - A_S \rvert$
5	0.0103	**0.004630**
6	0.0126	**0.011221**
7	0.0084	**0.007625**
8	0.0009	**0.000031**
9	0.0047	**0.004254**
10	0.0036	**0.003158**
11	0.0014	**0.000798**
12	0.0033	**0.003039**
13	0.0012	**0.000657**
14	0.0017	**0.001347**
15	0.0021	**0.001769**
16	0.0011	**0.000650**
17	0.0019	**0.001452**
18	0.0010	**0.000722**
19	0.0011	**0.000852**
20	0.0012	**0.000922**

The results of Table 1 revealed that the saddlepoint approximation to the distribution of the V statistic is more suitable than the Bagai's approximation at a 1% significance level. For the cases of $n = 8$, 9 and 15, Bagai's approximation is better than the saddlepoint approximation. However, the saddlepoint approximation is conservative for the exact probability of the V statistic for $n = 8$ and 15. In addition, Table 2 indicates that the saddlepoint approximation to the distribution of the V statistic is better than the Bagai's approximation at the 5% significance level. We then estimated the exact critical values of the Bagai statistic using the approximation A_S for large sample sizes because it is difficult to derive an exact critical value otherwise.

Table 3: Critical values of the V statistic by saddlepoint approximation.

n	v	Probability	v	Probability
21	323	0.00991	231	0.05002
22	347	0.00997	248	0.05017
23	372	0.00994	266	0.04992
24	397	0.01001	284	0.04994
25	423	0.01000	302	0.05018
30	561	0.00997	400	0.04999
35	711	0.00998	506	0.05012
40	872	0.01001	621	0.04993
45	1044	0.01001	743	0.04993
50	1226	0.01002	872	0.04994

4. ORDERS OF ERRORS OF SADDLEPOINT APPROXIMATION

In this section, we consider the error orders of a saddlepoint approximation. From Section 3, we developed a standardized cumulant generating function of the V statistic as follows:

$$\kappa_*(s) = -n\log 2 - \frac{3\,n\,(n-1)\,s}{2\,\sigma} + \sum_{j=2}^{n+1} \log\Big\{1 + \exp\big(2\,s\,(2n-j)/\sigma\big)\Big\} ,$$

where

$$\sigma^2 = \frac{n\,(n-1)\,(14\,n-13)}{6} .$$

Then, we obtained the first four derivatives of the standardized cumulant generating as follows:

$$\kappa_*^{(1)}(s) = -\frac{3\,n\,(n-1)}{2\,\sigma} + \sum_{j=2}^{n+1} \frac{C_j \exp(s\,C_j)}{1+\exp(s\,C_j)} ,$$

$$\kappa_*^{(2)}(s) \;=\; \sum_{j=2}^{n+1} \frac{C_j^2 \exp(s\,C_j)}{\big\{1+\exp(s\,C_j)\big\}^2}\;,$$

$$\kappa_*^{(3)}(s) \;=\; \sum_{j=2}^{n+1} \frac{C_j^3 \exp(s\,C_j)\big\{1-\exp(s\,C_j)\big\}}{\big\{1+\exp(s\,C_j)\big\}^3}$$

and

$$\kappa_*^{(4)}(s) \;=\; \sum_{j=2}^{n+1} \frac{C_j^4 \exp(s\,C_j)\big\{1-4\exp(s\,C_j)+\exp(2\,s\,C_j)\big\}}{\big\{1+\exp(s\,C_j)\big\}^4}\;,$$

where

$$C_j \;=\; \frac{2\,(2\,n-j)}{\sigma}\;.$$

The standardized skewness and standardized kurtosis was then given by

$$\text{standardized skewness:}\quad \frac{\kappa_*^{(3)}(0)}{\kappa_*^{(2)}(0)^{3/2}} = 0$$

$$\text{(4.1)}\quad \text{standardized kurtosis:}\quad \frac{\kappa_*^{(4)}(0)}{\kappa_*^{(2)}(0)^{2}} = \frac{-12\big(186\,n^3-489\,n^2+421n-119\big)}{5\,n\,(n-1)\,(14n-13)^2}\;.$$

Bagai *et al.* noted that the normal approximation was appropriate for $n>20$ but that the difference of the standardized kurtosis from zero is (4.1).

We next derived the orders of the errors of the V statistic. By using an expansion for the standardized cumulant generating function, we approximated the $\kappa_*(s)$ as follows:

$$\begin{aligned}
\kappa_*(s) \;\approx\;& -n\log 2-\frac{3\,n\,(n-1)\,s}{2\,\sigma}+\sum_{j=2}^{n+1}\left\{\log 2+\frac{2\,s\,(2n-j)}{2\,\sigma}+\frac{4s^2\,(2n-j)^2}{8\,\sigma^2}\right.\\
&\left.-\frac{16s^4\,(2n-j)^4}{192\,\sigma^4}+\frac{64s^6\,(2n-j)^6}{2880\,\sigma^6}+\cdots\right\}\\
\approx\;& \frac{ns^2\,(n-1)\,(14n-13)}{12\,\sigma^2}-\frac{ns^4\,(n-1)\,(186\,n^3-489\,n^2+421n-119)}{360\,\sigma^4}\\
&+\frac{ns^6\,(n-1)\,(76\,n^5-3207n^4+5256\,n^3-494n^2+1637n-253)}{1890\,\sigma^6}\\
\approx\;& \frac{s^2}{2}-\frac{1}{n}\left\{\frac{s^4\big(186\,n^3-489\,n^2+421n-119\big)}{10\,(n-1)\,(14n-13)^2}\right\}\\
&+\frac{1}{n^2}\left\{\frac{4s^6\big(762\,n^5-3207n^4+5256\,n^3-4194n^2+1637n-253\big)}{35\,(n-1)^2\,(14n-13)^3}\right\}\\
&+O(n^{-3})\;.
\end{aligned}$$

We then approximated the first four derivatives of the standardized cumulant generating function by

$$\kappa_*^{(1)}(s) \approx s - \frac{1}{n}\left\{\frac{2s^3\left(186\,n^3-489\,n^2+421n-119\right)}{5(n-1)(14n-13)^2}\right\} + \frac{1}{n^2}\left\{\frac{24s^5\left(762\,n^5-3207n^4+5256\,n^3-4194n^2+1637n-253\right)}{35(n-1)^2(14n-13)^3}\right\} + O(n^{-3}) \,,$$

$$\kappa_*^{(2)}(s) \approx 1 - \frac{1}{n}\left\{\frac{6s^2\left(186\,n^3-489\,n^2+421n-119\right)}{5(n-1)(14n-13)^2}\right\} + \frac{1}{n^2}\left\{\frac{24s^4\left(762\,n^5-3207n^4+5256\,n^3-4194n^2+1637n-253\right)}{7(n-1)^2(14n-13)^3}\right\} + O(n^{-3}) \,,$$

$$\kappa_*^{(3)}(s) \approx -\frac{1}{n}\left\{\frac{12s\left(186\,n^3-489\,n^2+421n-119\right)}{5(n-1)(14n-13)^2}\right\} + \frac{1}{n^2}\left\{\frac{96s^3\left(762\,n^5-3207n^4+5256\,n^3-4194n^2+1637n-253\right)}{7(n-1)^2(14n-13)^3}\right\} + O(n^{-3})$$

and

$$\kappa_*^{(4)}(s) \approx -\frac{1}{n}\left\{\frac{12\left(186\,n^3-489\,n^2+421n-119\right)}{5(n-1)(14n-13)^2}\right\} + \frac{1}{n^2}\left\{\frac{288\,s^2\left(762\,n^5-3207n^4+5256\,n^3-4194n^2+1637n-253\right)}{7(n-1)^2(14n-13)^3}\right\} + O(n^{-3}) \,.$$

By expanding for $\kappa_*(s)$ in a Taylor series, we then determined

$$0 = \kappa_*(0) = \kappa_*(s) - s\kappa_*^{(1)}(s) + \frac{s^2\kappa_*^{(2)}(s)}{2} - \frac{s^3\kappa_*^{(3)}(s)}{6} + \frac{s^4\kappa_*^{(4)}(s)}{24} + \cdots \,.$$

Then, substituting to the w, we determined

$$\begin{aligned} w^2 &= 2\left\{s\kappa_*^{(1)}(s) - \kappa_*(s)\right\} \\ &= 2\left\{s\kappa_*^{(1)}(s) - s\kappa_*^{(1)}(s) + \frac{s^2\kappa_*^{(2)}(s)}{2} - \frac{s^3\kappa_*^{(3)}(s)}{6} + \frac{s^4\kappa_*^{(4)}(s)}{24} + \cdots\right\} \\ &= s^2\kappa_*^{(2)}(s)\left\{1 - \frac{s\kappa_*^{(3)}(s)}{3\,\kappa_*^{(2)}(s)} + \frac{s^2\kappa_*^{(4)}(s)}{12\,\kappa_*^{(2)}(s)} + \cdots\right\}. \end{aligned}$$

Therefore, we obtained

$$\begin{aligned}\frac{1}{w} &= \frac{1}{s\sqrt{\kappa_*^{(2)}(s)}}\left\{1-\frac{s\,\kappa_*^{(3)}(s)}{3\kappa_*^{(2)}(s)}+\frac{s^2\kappa_*^{(4)}(s)}{12\,\kappa_*^{(2)}(s)}+\cdots\right\}^{-\frac{1}{2}}\\ &= \frac{1}{s\sqrt{\kappa_*^{(2)}(s)}}\left\{1+\frac{s\,\kappa_*^{(3)}(s)}{6\kappa_*^{(2)}(s)}-\frac{s^2\kappa_*^{(4)}(s)}{24\,\kappa_*^{(2)}(s)}+\frac{s^2\big(\kappa_*^{(3)}(s)\big)^2}{24\big(\kappa_*^{(2)}(s)\big)^2}+\cdots\right\}\\ &= \frac{1}{s\sqrt{\kappa_*^{(2)}(s)}}+O(n^{-1})\end{aligned}$$

by applying the binomial theorem and substituting the i-th standardized cumulant. To determine the saddlepoint approximation to $\Pr\big((V-\mathrm{E}(V))/\sigma \geq v_*\big)$, we solved the saddlepoint equation, $\kappa_*^{(1)}(s)=v_*$, and used the unique solution $(s=\hat{s})$ to calculate

$$\hat{w} = \sqrt{2\left(\hat{s}\,v_* - \kappa_*(\hat{s})\right)}\ \mathrm{sgn}(\hat{s}) \qquad \text{and} \qquad \hat{u} = \hat{s}\sqrt{\kappa_*^{(2)}(\hat{s})}\ ,$$

where $\mathrm{sgn}(\hat{s}) = \pm 1, 0$ if $\hat{s}$ is positive, negative, or zero. Therefore, we determined

$$\begin{aligned}\Pr\left(\frac{V-\mathrm{E}(V)}{\sigma}\geq v_*\right) &\approx 1-\Phi(\hat{w})+\phi(\hat{w})\left\{\frac{1}{\hat{u}}-\frac{1}{\hat{w}}+O(n^{-1})\right\}\\ &= 1-\Phi(\hat{w})+\phi(\hat{w})\left(\frac{1}{\hat{u}}-\frac{1}{\hat{w}}\right)+O(n^{-1})\ .\end{aligned}$$

Note that $\phi(\hat{w}) \approx \text{Constant} + O(n^{-1})$. Typically, an approximation of the above form has the relative error $O(n^{-3/2})$. However, the exact distribution of the standardized Bagai statistic is discrete, so the discrete distribution of the standardized Bagai statistic may be approximated at its support point by a smooth function that behaves similarly to a distribution function. Therefore, $\Phi(r^*(w))$ approximates the distribution function of the Bagai statistic with a relative error of $O(n^{-1})$ in a normal deviation region in which $r^*(w) = w + w^{-1}\log(u/w)$; Barndorff-Nielsen and Cox (1994). The use of saddlepoint approximation as a technique for smoothing discrete distributions is discussed by Davison and Wang (2002).

5. CONCLUDING REMARKS

In this paper, we considered the saddlepoint approximation to the distribution of the Bagai statistic V (1989). The standard saddlepoint formula provided an accurate approximation to the distribution of the V statistic. From the numerical results, we determined that the approximation precision of the saddlepoint approximation is superior to the Bagai's approximation using finite sample sizes. The orders of the errors of a saddlepoint approximation were also derived. In future work, we intend to 1) compare the orders of the errors of the higher-order saddlepoint approximation, Bagai's approximation, and other approximations, 2) be able to apply the saddlepoint approximation to other statistics for testing the independent competing risks model, and 3) consider the saddlepoint approximation to the distribution of the V statistic for cases of dependent competing risks models, *e.g.* Aly, Kochar and McKeague (1994), Dykstra, Kochar and Robertson (1995).

ACKNOWLEDGMENTS

The author would like to give his thank to the referee and the editor for their valuable comments and suggestions. It is a pleasure to acknowledge conversations with Professor T. Yanagimoto.

REFERENCES

[1] Aly, E.A.A.; Kochar, S.C. and McKeague, L.W. (1994). Some tests for comparing cumulative incidence functions and cause-specific hazard rates, *Journal of the American Statistical Association*, **89**, 994–999.

[2] Anderson, T.W. and Darling, D.A. (1952). Asymptotic theory of certain goodness-of-fit criteria based on stochastic processes, *Annals of Mathematical Statistics*, **23**, 193–212.

[3] Anderson, T.W. and Darling, D.A. (1954). A test of goodness of fit, *Journal of the American Statistical Association*, **49**, 765–769.

[4] Bagai, I.; Deshpandé, J.V. and Kochar, S.C. (1989). Distribution free tests for stochastic ordering in the competing risks model, *Biometrika*, **76**(4), 775–781.

[5] Barndorff-Nielsen, O.E. and Cox, D.R. (1994). *Inference and Asymptotic*, Chapman and Hall.

[6] Bean, R.; Froda, S. and van Eeden, C. (2004). The normal, Edgeworth, saddlepoint and uniform approximations to the Wilcoxon–Mann–Whitney null-distribution: A numerical comparison, *Journal of Nonparametric Statistics*, **16**(1–2), 279–288.

[7] Butler, R.W. (2007). *Saddlepoint Approximations with Applications*, Cambridge University Press.

[8] Chen, Q. and Giles, D.E. (2008). General saddlepoint approximations: Application to the Anderson–Darling test statistic, *Communications in Statistics – Simulation and Computation*, **37**, 789–804.

[9] Daniels, H.E. (1954). Saddlepoint approximations in statistics, *Annals of Mathematical Statistics*, **25**, 631–650.

[10] Daniels, H.E. (1987). Tail probability approximations, *International Statistical Review*, **55**, 37–48.

[11] Davison, A.C. and Wang, S. (2002). Saddlepoint approximations as smoothers, *Biometrika*, **89**(4), 933–938.

[12] Dykstra, R.; Kochar, S. and Robertson, T. (1995). Likelihood based inference for cause specific hazard rates under restrictions, *Journal of Multivariate Analysis*, **54**, 163–174.

[13] Easton, G.S. and Ronchetti, E. (1986). General saddlepoint approximations with applications to L statistics, *Journal of the American Statistical Association*, **81**, 420–430.

[14] Froda, S. and van Eeden, C. (2000). A uniform saddlepoint expansion for the null-distribution of the Wilcoxon–Mann–Whitney statistic, *Canadian Journal of Statistics*, **28**, 137–149.

[15] Gibbons, J.D. and Chakraborti, S. (2003). *Nonparametric Statistical Inference*, Fourth Edition, Dekker, New York.

[16] Giles, D.A.E. (2001). A saddlepoint approximation to the distribution function of the Anderson–Darling test statistic, *Communications in Statistics – Simulation and Computation*, **30**, 899–905.

[17] Goutis, C. and Casella, G. (1999). Explaining the saddlepoint approximation, *American Statistician*, **53**, 216–224.

[18] Hu, X.S. and Tsai, W.Y. (1999). Linear rank tests for competing risks model, *Statistica Sinica*, **9**, 971–983.

[19] Huzurbazar, S. (1999). Practical saddlepoint approximations, *American Statistician*, **53**, 225–232.

[20] Jensen, J.L. (1995). *Saddlepoint Approximations*, Oxford University Press.

[21] Kolassa, J.E. (2006). *Series Approximation Methods in Statistics*, Springer-Verlag.

[22] Lugannani, R. and Rice, S.O. (1980). Saddlepoint approximation for the distribution of the sum of independent random variables, *Advances in Applied Probability*, **12**, 475–490.

[23] Neuhaus, G. (1991). Some linear and nonlinear rank tests for competing risks models, *Communications in Statistics – Theory and Methods*, **20**, 667–701.

[24] Reid, N. (1988). Saddlepoint methods and statistical inference (with discussion), *Statistical Science*, **3**, 213–238.

[25] Sinclair, C.D. and Spurr, B.D. (1988). Approximations to the distribution function of the Anderson–Darling test statistic, *Journal of the American Statistical Association*, **83**, 1190–1191.

[26] Wood, A.T.A.; Booth, J.G. and Butler, R.W. (1993). Saddlepoint approximations to the CDF of some statistics with nonnormal limit distributions, *Journal of the American Statistical Association*, **88**, 680–686.

[27] Yip, P. and Lam, K.F. (1992). A class of non-parametric tests for the equality of failure rates in a competing risks model, *Communications in Statistics – Theory and Methods*, **21**, 2541–2556.