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ESTIMATING OF THE PROPORTIONAL HAZARD PREMIUM FOR HEAVY-TAILED CLAIM AMOUNTS WITH THE POT METHOD

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Abstract:

- In this paper we propose a new estimator of the proportional hazard premium for heavy-tailed claim amounts, with a help of the peak-over-threshold (POT) method. We establish the asymptotic normality of the new estimator, and its performance is illustrated in a simulation study. Moreover, we compare, in terms of bias and mean squared error, our estimator with the estimator of Necir and Meraghni (2009).

Key-Words:

- *distortion risk measures; proportional hazard premium; extreme values; GPD function; heavy tails; POT method.*

AMS Subject Classification:

- 62G32, 31B30.

1. INTRODUCTION AND MOTIVATION

A general class for constructing loaded pricing functional, introduced in the actuarial literature by Wang (1996), is namely the Distortion Risk Measures (DRM). For a given nondecreasing function $g\colon [0,1] \to [0,1]$ such that $g(0) = 0$ and $g(1) = 1$, for any nonnegative random variable X, where X is an insured risk with distribution function (df) F, and the tail of F will be denoted by $\overline{F} = 1 - F$. The distorted expectation is defined as follows:

$$\Pi_\rho = \int_0^\infty g\big(\overline{F}(x)\big)\, dx \; . \tag{1.1}$$

The function g is called a distortion function, if g is concave the DRM further satisfies the subadditivity and becomes coherent in the sense of Artzner *et al.* (1999); see, e.g., Wirch and Hardy (2000) and Dhaene *et al.* (2006). Some examples of continuous concave distortion functions corresponding to familiar risk measures are presented below, by choosing a suitable function g, one can easily express some popular risk measures:

The Tail-VaR: $g(x) = x/(1-q) \wedge 1$, $q \in (0,1)$,
Proportional Hazard Transform: $g(x) = x^{1/\rho}$, $\rho \geq 1$,
Dual-Power Transform: $g(x) = 1 - (1-x)^\rho$, $\rho > 1$,
Wang Transform: $g(x) = \Phi\big(\Phi^{-1}(x) - \Phi^{-1}(\rho)\big)$,
where $\Phi(\cdot)$ is the df of the standard normal.

In this paper, we are interested by estimate the proportional hazard transform, that is

$$\Pi_\rho = \int_0^\infty \big(\overline{F}(x)\big)^{1/\rho}\, dx \; , \tag{1.2}$$

where $\rho \geq 1$ represents the distortion coefficient or the risk aversion index.

In practice the estimation of these risk measures from a sample of rv's i.i.d. $X_1, X_2, ..., X_n$, are based on the empirical distribution function F_n. The asymptotic behavior of this estimator has been studied by Jones and Zitikis (2003) provided that, the second moment is finite.

Now, assume that F is heavy tailed. This class includes popular distributions (such as Pareto, Burr, Student, Lévy-stable and log-gamma) known to be very appropriate models for fitting large insurance claims, large fluctuations of prices, log-returns and other data (see for instance, Beirlant *et al.*, 2001). In the remainder of the paper, we restrict ourselves to this class, more specifically, we deal within the class of regularly varying cdf's. For more details on this type of distributions, we refer to Bingham *et al.* (1987) and Rolski *et al.* (1999).

The tail of F is said to be with regulary varying at infinity, if

$$\overline{F}(x) = c\,x^{-1/\xi}\big(1+x^{-\delta}\,\mathbb{L}(x)\big), \qquad \text{as} \quad x\to\infty\,, \tag{1.3}$$

for $\xi\in(0,1)$, $\delta>0$ and some real constant c, with $\mathbb{L}$ a slowly varying function, i.e. $\mathbb{L}(tx)/\mathbb{L}(x)\to 1$ as $x\to\infty$ for any $t>0$. For further properties of these functions, see chapter 0 in Resnick (1987) or Seneta (1976).

For example, when the tail of df is Pareto with $\xi=3/4$, we have $\mathbb{E}(X^2)=\infty$. To solve this problem, Necir and Meraghni (2009) used the extreme values approach and propose an asymptotically normal semiparametric estimator for Π_ρ. This estimator is based on the extreme quantile estimator of Weissman (1978). However this quantile is biased.

In this paper, we use the result of Balkema and de Haan (1974) and Pickands (1975), which states that for a certain class of distributions the Generalised Pareto Distribution (GPD) is the limiting for the distribution of the excesses F_u, as the threshold u tends to the right endpoint y_F. Formally, we can find a positive measurable function $\beta(u)$, such as

$$\lim_{u\to y_F}\ \sup_{0<y<y_F-u}\big|F_u(y)-\mathbb{G}_{\xi,\beta(u)}(y)\big| = O\big(u^{-\delta}\,\mathbb{L}(u)\big)\,, \tag{1.4}$$

where $u^{-\delta}\,\mathbb{L}(u)\to 0$ as $u\to\infty$, for any $\delta>0$.

We investigate this result for purpose a alternative estimator for the proportional hazard transform Π_ρ, as follows:

$$\widehat{\Pi}_{\rho,n} = \int_0^{u_n}\bigg(n^{-1}\sum_{j=1}^{n}\mathbf{1}\big(X_j\ge x\big)\bigg)^{1/\rho}dx \,+\, (\widehat{p}_n)^{1/\rho}\frac{\rho\,\widehat{\beta}_n}{1-\widehat{\xi}_n\,\rho}\,. \tag{1.5}$$

Under suitable assumptions, this estimator are asymptotically normal distributed and unbiased with an easily estimated variance.

The paper is organized as follows. In the second section of the paper, the new estimator of Π_ρ is introduced and its properties examined. This is followed by a simulation study of its behavior in comparison with the Necir and Meraghni estimator. Finally, the proofs of our result are postponed until the last section.

2. DEFINING THE ESTIMATOR AND THE MAIN RESULT

Let $X_1,...,X_n$ be an independent and identically distributed random variables, each with the same cdf F, and let u_n be some a large number, 'high level', which we later let tends to infinity when $n\to\infty$. With the notation

$$\overline{F}_{u_n}(y) = P\Big(X_1-u_n>y \mid X_1>u_n\Big)\,,$$

we have

$$\overline{F}_{u_n}(y) \;=\; \overline{F}(u_n+y)/\overline{F}(u_n)\ ,$$

and thus

$$\overline{F}_{u_n}(y) \;=\; \left(1+\frac{y}{u_n}\right)^{-1/\xi} \frac{1+(u_n+y)^{-\delta}\,\mathbb{L}(u_n+y)}{1+u_n^{-\delta}\,\mathbb{L}(u_n)}\ ,$$

and if $\beta=u_n\xi$, then $\overline{F}_{u_n}(y)$ is a GPD perturbed, where the df of the GPD has the form

$$\mathbb{G}_{\xi,\beta}(y) = \begin{cases} 1-\left(1+\xi\frac{y}{\beta}\right)^{-\frac{1}{\xi}}, & \xi\neq 0, \quad 0\leq y<\infty \ \text{ if } \xi\geq 0\ , \\ 1-\exp\left(-y/\beta\right), & \xi=0, \quad 0\leq y<-\beta/\xi \ \text{ if } \xi<0\ . \end{cases} \tag{2.1}$$

This means that, with the result (1.4) of Balkema and de Haan (1974) and Pickands (1975), for large values of u_n, we have

$$F_{u_n}(y) \;\approx\; \mathbb{G}_{\xi,\beta(u_n)}(y)\ . \tag{2.2}$$

By the definition of the excess distribution, we have

$$\overline{F}(u_n+t) \;=\; \overline{F}(u_n)\,\overline{F}_{u_n}(t)\ ,$$

and, denote by

$$N=N_{u_n} \;=\; \text{card}\Big\{X_i>u_n : 1\leq i\leq n\Big\}\ ,$$

the number of exceedance over u_n, we have $N \rightsquigarrow \mathcal{B}(p_n,n)$, where $p_n=P(X_1>u_n)$. A natural estimator for $p_n=\overline{F}(u_n)$ is $\widehat{p}_n=N/n$. Let

$$Y_{i,n}=X_j-u_n\ , \qquad \text{provided} \quad X_j>u_n, \quad i=1,...,N\ ,$$

(where j is the index of the i^{th} exceedance) are i.i.d. rv's with cdf F_{u_n} based on the sample $(Y_{1:n},Y_{2:n},...,Y_{N_n,n})$, the approximation (2.2) motivates us to take an estimator for $\overline{F}_{u_n}(y)$ as follows:

$$\widehat{\overline{F}}_{u_n}(y)=\overline{\mathbb{G}}_{\widehat{\xi}_n,\widehat{\beta}_n}(y)\ , \qquad y>0\ . \tag{2.3}$$

Therefore, an estimator of $\overline{F}(u_n+y)$ is

$$\widehat{\overline{F}}(u_n+y) \;=\; \widehat{\overline{F}}(u_n)\,\widehat{\overline{F}}_{u_n}(y) \;=\; \widehat{p}_n\,\overline{\mathbb{G}}_{\widehat{\xi}_n,\widehat{\beta}_n}(y)\ , \tag{2.4}$$

where $\widehat{\xi}_N$ and $\widehat{\beta}_N$ are consistent estimators of ξ and β respectively. Moreover, these estimators are asymptotically normal provided that $\xi>-1/2$. Smith (1987) established in theorem (3.2), the asymptotic normality of $(\widehat{\xi}_N,\widehat{\beta}_N)$ as follows:

$$\sqrt{N}\begin{pmatrix} \widehat{\beta}_N/\beta_N-1 \\ \widehat{\xi}_N-\xi \end{pmatrix} \overset{\mathcal{D}}{\to} \mathcal{N}_2(0,\mathbb{Q}^{-1}) \qquad \text{as} \quad N\to\infty\ , \tag{2.5}$$

where

$$\mathbb{Q}^{-1} = (1+\xi)\begin{pmatrix} 2 & -1 \\ -1 & 1+\xi \end{pmatrix}, \tag{2.6}$$

provided that $\sqrt{N}u_N^{-\delta}\,\mathbb{L}(u_N) \to 0$ as $N \to \infty$ and $x \to x^{-\delta}\mathbb{L}(x)$ is non-increasing near infinity. In the case $\sqrt{N}u_N^{-\delta}\,\mathbb{L}(u_N) \nrightarrow 0$, the limiting distribution in (2.5) is biased.
Here $\overset{\mathcal{D}}{\to}$ denotes convergence in distribution and $\mathcal{N}_2(0,\epsilon^2)$ stands for the normal r.v. of mean 0 and variance ϵ^2.

We assume that the tail of the distribution start at the threshold u_n, then, we have

$$\Pi_\rho = \int_0^{u_n} \big(\overline{F}(x)\big)^{1/\rho} dx + \int_{u_n}^{\infty} \big(\overline{F}(x)\big)^{1/\rho} dx\ , \qquad \rho > 1\ . \tag{2.7}$$

An estimator of Π_ρ is given by replacing (2.4) in equation (2.7), as follows:

$$\widehat{\Pi}_{\rho,n}(x) = \int_0^{u_n} \big(\overline{F}_n(x)\big)^{1/\rho} dx + (\widehat{p}_n)^{1/\rho} \int_0^{\infty} \big(\overline{\mathbb{G}}_{\widehat{\xi}_n,\widehat{\beta}_n}(y)\big)^{1/\rho} dy\ ,$$

where F_n is the empirical distribution function pertaining to the sample $X_1, X_2, ...,$ X_n. After Integration, we obtain the new estimator given by formula (1.5).

The asymptotic normality of $\widehat{\Pi}_{\rho,n}$ is established in the following theorem.

Theorem 2.1. *Let F be a distribution function fulfilling (1.3) with $\xi \in (1/2,1)$. Suppose that the function $\mathbb{L}$ is locally bounded in $[x_0,+\infty)$ for $x_0 \geq 0$ and $x \to x^{-\delta}\mathbb{L}(x)$ is non-increasing near infinity, for some $\delta > 0$. For any $u_n = O(n^{\alpha\xi})$ with $\alpha \in (0,1)$, and $\rho > 1$ such that $4\alpha/\rho - 2\alpha\,\xi < 1$, we have*

$$\frac{\sqrt{n}}{\gamma_n \sigma_n}\big(\widehat{\Pi}_{\rho,n} - \Pi_\rho\big) \overset{\mathcal{D}}{\to} \mathcal{N}(0,1)\,, \qquad \text{as} \quad n \to \infty\ ,$$

where

$$\sigma_n^2 := \frac{1}{\rho^2} + \frac{\theta_1^2}{\gamma_n^2}\,p_n(1-p_n) + \frac{2(1+\xi)\,\theta_2^2\,\beta_n^2}{p_n\gamma_n^2} + \frac{(1+\xi)^2\,\theta_3^2}{p_n\gamma_n^2} - \frac{(1+\xi)\,\beta_n\,\theta_2\,\theta_3}{p_n\gamma_n^2}\ ,$$

$$\gamma_n^2 = \operatorname{var}\left(\int_0^{u_n} \big(\overline{F}(x)\big)^{1/\rho-1} \mathbf{1}_{\{X_1 \leq x\}}\,dx\right), \tag{2.8}$$

and

$$\theta_1 = \frac{\beta_n(p_n)^{1/\rho-1}}{1-\xi\rho}\ , \qquad \theta_2 = \frac{\rho(p_n)^{1/\rho}}{1-\xi\rho}\ , \qquad \theta_3 = \frac{\rho^2\beta_n(p_n)^{1/\rho}}{(1-\xi\rho)^2}\ ,$$

with $\beta_n = u_n\xi$.

3. SIMULATION STUDY

In this section, we carry out a simulation study (by means of the statistical software **R**, see Ihaka and Gentleman, 1996) to illustrate the performance of our estimation procedure and its comparison with the estimator of Necir and Meraghni (2009). We generate samples from Fréchet distributions with tail $\overline{F}(x) = 1 - \exp(-x^{-1/\xi})$, $x > 0$ (with tail index $\xi = 2/3$ and $\xi = 3/4$) and two distinct aversion index values $\rho = 1.1$ and $\rho = 1.2$.

In the first part, we evaluate the accuracy of the confidence intervals via their lengths and coverage probabilities (cov prob), we generate 200 independent replicates of sizes 1000 and 2000 from the selected parent distribution. For each simulated sample, we obtain a value of the estimators premium Π_ρ. The overall estimated premium Π_ρ is then taken as the empirical mean of the values in the 200 repetitions. We summarize the results in Table 1.

Table 1: Point estimates and 95%-confidence intervals for Π, based on 200 samples of Fréchet distributed rv's with tail index 2/3 and 3/4 with aversion index 1.1 and 1.2.

ξ	2/3				3/4			
ρ	1.1		1.2		1.1		1.2	
Π	3.439		4.699		5.351		9.645	
n	1000	2000	1000	2000	1000	2000	1000	2000
$\widehat{\Pi}_{\rho,n}$	3.775	3.571	4.517	4.619	5.450	5.441	9.439	9.692
rmse	0.570	0.515	0.781	0.639	0.187	0.127	0.109	0.107
lcb	2.661	2.896	2.166	3.803	3.567	3.744	5.921	6.909
ucb	4.482	4.655	6.867	6.434	7.333	7.139	12.957	12.476
length	1.821	1.759	4.701	2.631	3.765	3.395	7.036	5.567
cprob	0.785	0.815	0.75	0.821	0.975	0.98	0.85	0.85

In the second part in this study, we generate 200 independent replicates of sizes 1000 from a Fréchet distribution, we compare the bias and the root mean squared error (RMSE) of the two estimators of Π_ρ (our estimator $\widehat{\Pi}_{\rho,n}$ with the estimator of Necir and Meraghni $\widetilde{\Pi}_{\rho,n}$). The results are presented in Table 2.

Table 2: Analog between the new estimator and the estimator of Necir and Meraghni for the premium hazard proportional for two tail index and two risk aversions index.

ξ	2/3		3/4	
ρ	1.1	1.2	1.1	1.2
Π_ρ	3.44	4.699	5.350	9.645
$\widehat{\Pi}_{\rho,n}$	3.527	4.807	5.359	9.499
bias	0.087	0.108	0.009	0.142
RMSE	0.335	0.592	0.516	0.933
$\widetilde{\Pi}_{\rho,n}$	4.221	4.938	5.452	9.915
bias	0.781	0.238	0.136	0.262
RMSE	0.867	0.665	0.674	1.131

4. PROOF OF THE MAIN RESULT

The following proposition is instrumental for the proof of our result

Proposition 4.1. *Let F be a distribution function fulfilling (1.3) with $\xi \in (0,1)$, $\delta > 0$ and some real c. Suppose that $\mathbb{L}$ is locally bounded in $[x_0, +\infty)$ for $x_0 \geq 0$. Then, for n large enough, for any $u_n = O(n^{\alpha\xi})$, $\alpha \in (0,1)$, we have*

$$p_n = P\big(X_1 > u_n\big) = c\big(1 + o(1)\big)\, n^{-\alpha}\,,$$

$$\gamma_n^2 = \operatorname{var}\!\left(\int_0^{u_n} \big(\overline{F}(x)\big)^{1/\rho-1} \mathbf{1}_{\{X_1 \leq x\}}\, dx\right) = O\big(n^{2\alpha(\xi-1/\rho+1)}\big)\,,$$

and

$$\sqrt{np_n}\, u_n^{-\delta}\, \mathbb{L}(u_n) = O\big(n^{-\alpha/2-\alpha\xi\delta+1/2}\big)\,.$$

Proof of the Theorem 2.1: Let us write

$$\sqrt{n}\,\big(\widehat{\Pi}_{\rho,n} - \Pi_\rho\big) = A_n + B_n\,, \tag{4.1}$$

where

$$A_n = \sqrt{n} \int_0^{u_n} \Big[\big(\overline{F}_n(x)\big)^{1/\rho} - \big(\overline{F}(x)\big)^{1/\rho}\Big]\, dx\,,$$

and

$$B_n = \sqrt{n}\left(\widehat{p}_n^{1/\rho} \frac{\rho\,\widehat{\beta}_n}{1 - \widehat{\xi}_n\,\rho} - \int_{u_n}^{\infty} \big(\overline{F}(x)\big)^{1/\rho}\, dx\right).$$

We begin by B_n, we may rewrite B_n as follows:

$$B_n = B_{n,1} + B_{n,2}\,,$$

where

$$B_{n,1} \;=\; (\widehat{p}_n)^{1/\rho}\,\frac{\rho\,\widehat{\beta}_n}{1-\widehat{\xi}_n\,\rho} \;-\; (p_n)^{1/\rho}\,\frac{\rho\,\beta_n}{1-\xi\,\rho}\;,$$

and

$$B_{n,2} \;=\; (p_n)^{1/\rho}\,\frac{\rho\,\beta_n}{1-\xi\rho} \;-\int_{u_n}^{\infty}\big(\overline{F}(s)\big)^{1/\rho}\,ds\;.$$

First, observe that $B_{n,1}$, may be rewrite into

$$\begin{aligned} B_{n,1} \;=\; & \frac{\rho\,\widehat{\beta}_n}{1-\widehat{\xi}_n\,\rho}\,\sqrt{n}\Big((\widehat{p}_n)^{1/\rho}-(p_n)^{1/\rho}\Big) \\ & +\,(p_n)^{1/\rho}\,\frac{\rho}{1-\widehat{\xi}_n\,\rho}\,\sqrt{n}\,\big(\widehat{\beta}_n-\beta_n\big) \\ & +\,\frac{\rho^2\,\beta_n(p_n)^{1/\rho}}{(1-\widehat{\xi}_n\,\rho)\,(1-\xi\,\rho)}\,\sqrt{n}\,\big(\widehat{\xi}_n-\xi\big)\;. \end{aligned}$$

From Smith (1987), we have, as $n\to\infty$

$$\widehat{\beta}_n/\beta_n - 1 \;=\; O_{\mathbb{P}}\big(u_n^{-\delta}\,\mathbb{L}(u_n)\big) \qquad\text{and}\qquad \widehat{\xi}_n-\xi \;=\; O_{\mathbb{P}}\big(u_n^{-\delta}\,\mathbb{L}(u_n)\big)\;. \tag{4.2}$$

On the other hand, by the central limit theorem, we have

$$\widehat{p}_n - p_n = O_{\mathbb{P}}\big(\sqrt{p_n/n}\,\big) \qquad \text{as} \quad n\to\infty\;. \tag{4.3}$$

Then, with the delta method, we obtain

$$\begin{aligned} B_{n,1} \;=\; & \theta_1\big(1+o_{\mathbb{P}}(1)\big)\,\sqrt{n}\,(\widehat{p}_n-p_n) \\ & +\,\theta_2\big(1+o_{\mathbb{P}}(1)\big)\,\sqrt{n}\,(\widehat{\beta}_n-\beta_n) \\ & +\,\theta_3(1+o_{\mathbb{P}}(1))\,\sqrt{n}\,(\widehat{\xi}_n-\xi)\;, \end{aligned}$$

where

$$\theta_1 = \frac{\beta(p_n)^{1/\rho-1}}{1-\xi\,\rho}\;, \qquad \theta_2 \;=\; \frac{\rho(p_n)^{1/\rho}}{1-\xi\,\rho}\;, \qquad \theta_3 \;=\; \frac{\rho^2\,\beta(p_n)^{1/\rho}}{(1-\xi\,\rho)^2}\;.$$

Either, for $B_{n,2}$, we have

$$B_{n,2} \;=\; (p_n)^{1/\rho}\,\frac{\rho\beta_n}{\xi\,\rho-1} \;-\int_{u_n}^{\infty}\big(\overline{F}(s)\big)^{1/\rho}\,ds\;.$$

We may rewrite

$$\overline{F}_{u_n}(s) \;=\; \frac{\overline{F}(u_n+s)}{\overline{F}(u_n)} \;=\; \Big(1+\frac{s}{u_n}\Big)^{-1/\xi}\,\frac{1+(u_n+s)^{-\delta}\,\mathbb{L}(u_n+s)}{1+u_n^{-\delta}\,\mathbb{L}(u_n)}\;.$$

This allows us to rewrite

$$
\begin{aligned}
\int_0^\infty \big(\overline{F}(s+u_n)\big)^{1/\rho} ds \; = \\
&= \big(\overline{F}(u_n)\big)^{1/\rho} \int_0^\infty \big(\overline{F}_{u_n}(s)\big)^{1/\rho} ds \\
&= \big(\overline{F}(u_n)\big)^{1/\rho} \int_0^\infty \left[\left(1+\frac{s}{u_n}\right)^{-1/\xi} \frac{1+(u_n+s)^{-\delta}\,\mathbb{L}(u_n+s)}{1+u_n^{-\delta}\,\mathbb{L}(u_n)}\right]^{1/\rho} ds \\
&= p_n^{1/\rho}\left(\frac{1}{1+u_n^{-\delta}\,\mathbb{L}(u_n)}\right)^{1/\rho} \\
&\quad \times \int_0^\infty \left[\left(1+\frac{s}{u_n}\right)^{-1/\xi} \Big(1+(u_n+s)^{-\delta}\,\mathbb{L}(u_n+s)\Big)\right]^{1/\rho} ds \\
&= p_n^{1/\rho}\left(\frac{1}{1+u_n^{-\delta}\,\mathbb{L}(u_n)}\right)^{1/\rho} u_n^{1/\xi\rho} \int_{u_n}^\infty x^{-1/\xi\rho}\big(1+x^{-\delta}\,\mathbb{L}(x)\big)^{1/\rho} dx \\
&= p_n^{1/\rho}\left(\frac{1}{1+u_n^{-\delta}\,\mathbb{L}(u_n)}\right)^{1/\rho} u_n^{1/\xi\rho} \\
&\quad \times \left[\left(\frac{\xi\rho}{1-\xi\rho}\,u_n^{1-1/\xi\rho}\right) + \int_{u_n}^\infty x^{-1/\xi\rho-\delta}\,\mathbb{L}(x)^{1/\rho}\,dx\right].
\end{aligned}
$$

Since function $\mathbb{L}$ is locally bounded in $[x_0,\infty)$ for $x_0 \ge 0$ and $x^{-\delta}\mathbb{L}(x)$ is non-increasing near infinity, then for all large n, we have

$$
u_n^{1/\xi\rho} \int_{u_n}^\infty x^{-1/\xi\rho-\delta}\,\mathbb{L}(x)^{1/\rho}\,dx \;=\; O\big(u_n^{-\delta}\big)\;,
$$

and therefore, for all large n

$$
\int_{u_n}^\infty \overline{F}(x)^{1/\rho}\,dx \;=\; p_n^{1/\rho}\,\frac{\beta_n\,\rho}{1-\xi\rho}\Big(1-u_n^{-\delta}\,\mathbb{L}(u_n)+O\big(u_n^{-\delta}\,\mathbb{L}(u_n)\big)\Big)^{1/\rho}.
$$

Consequently

$$
B_{n,2} \;=\; O\big(u_n^{1-1/\rho\xi-\delta/\rho}\big)\;,
$$

which means, since $1-1/\rho\xi-\delta/\rho<0$, that $B_{n,2} \xrightarrow{P} 0$ as $n\to\infty$.
For A_n, we have

$$
A_n \;=\; \sqrt{n}\int_0^{u_n}\Big[\big(\overline{F}_n(x)\big)^{1/\rho}-\big(\overline{F}(x)\big)^{1/\rho}\Big]\,dx\;. \tag{4.4}
$$

We next show that, the right-hand side of (4.4), converge to 0 in probability, by the use of the Taylor formula, we have

$$\begin{aligned}
\int_0^{u_n} \Big[\big(\overline{F}_n(x)\big)^{1/\rho} - \big(\overline{F}(x)\big)^{1/\rho}\Big]\, dx \;=\;& \\
&= \frac{1}{\rho}\int_0^{u_n}\big(\overline{F}_n(x)-\overline{F}(x)\big)\,\big(\overline{F}(x)\big)^{1/\rho-1}\,dx \\
&= -\frac{1}{\rho}\int_0^{u_n}\big(F_n(x)-F(x)\big)\,\big(\overline{F}(x)\big)^{1/\rho-1}\,dx \\
&= -\frac{1}{\rho}\int_0^{u_n}\Big(\frac{1}{n}\sum \mathbf{1}(X_i\le x)-F(x)\Big)\,\big(\overline{F}(x)\big)^{1/\rho-1}\,dx \\
&= -\frac{1}{\rho}\left[\frac{1}{n}\sum\int_0^{u_n}\mathbf{1}(X_i\le x)\,\big(\overline{F}(x)\big)^{1/\rho-1}\,dx \;-\int_0^{u_n}F(x)\,\big(\overline{F}(x)\big)^{1/\rho-1}\,dx\right] \\
&= -\frac{1}{\rho}\Big[\overline{Z}-\mathbb{E}[Z_1]\Big]\;,
\end{aligned}$$

where

$$Z_i \;:=\; \int_0^{u_n}\big(\overline{F}(x)\big)^{1/\rho-1}\,\mathbf{1}(X_i\le x)\,dx\;.$$

We assume that

$$\gamma_n^2 \;=\; \mathrm{var}(Z_1)\;.$$

We are going to calculate γ_n. For $\overline{F}(x)=x^{-1/\xi}O(1)$ and $u_n=n^{\alpha\xi}O(1)$, we have

$$\begin{aligned}
\mathbb{E}[Z_i] \;&=\; \int_0^{u_n}\big(\overline{F}(x)\big)^{1/\rho-1}\,\mathbb{E}\big[\mathbf{1}(X_i\le x)\big]\,dx \\
&= \int_0^{u_n}\big(\overline{F}(x)\big)^{1/\rho-1}\,\big(1-\overline{F}(x)\big)\,dx \\
&= \int_0^{u_n}\big(\overline{F}(x)\big)^{1/\rho-1}dx \;-\int_0^{u_n}\big(\overline{F}(x)\big)^{1/\rho}dx \\
&= \left(\int_0^{u_n}\big(x^{-1/\xi(1/\rho-1)}\big)\,dx \;-\int_0^{u_n}\big(x^{-1/\xi\rho}\big)\,dx\right)O(1) \\
&= \left(\frac{\rho\,\xi\big(u_n^{1-1/\xi\rho+1/\xi}\big)}{\rho\,\xi+\xi-1}-\frac{\rho\,\xi\big(u_n^{1-1/\xi\rho}\big)}{\xi\rho-1}\right)O(1)
\end{aligned}$$

and

$$\begin{aligned}
E(Z_i^2) \;&=\; E\left[\int_0^{u_n}\big(\overline{F}(x)\big)^{1/\rho-1}\,\mathbf{1}(X_i\le x)\,dx\,\int_0^{u_n}\big(\overline{F}(y)\big)^{1/\rho-1}\,\mathbf{1}(X_i\le y)\,dy\right] \\
&= \left[\int_0^{u_n}\!\!\int_0^{u_n}\big(\overline{F}(x)\big)^{1/\rho-1}\,\big(\overline{F}(y)\big)^{1/\rho-1}\,E\Big[\mathbf{1}(X_i\le x)\,\mathbf{1}(X_i\le y)\Big]\,dx\,dy\right] \\
&= \left[\int_0^{u_n}\!\!\int_0^{u_n}\big(\overline{F}(x)\big)^{1/\rho-1}\,\big(\overline{F}(y)\big)^{1/\rho-1}\,\min\big(F(x),F(y)\big)\,dx\,dy\right] \;=
\end{aligned}$$

$$
\begin{aligned}
&= \int_0^{u_n} \big(\overline{F}(x)\big)^{1/\rho-1} \left(\int_0^{x} \big(\overline{F}(y)\big)^{1/\rho-1} F(y)\, dy \right) dx \\
&\quad + \int_0^{u_n} \big(\overline{F}(y)\big)^{1/\rho-1} \left(\int_x^{u_n} \big(\overline{F}(x)\big)^{1/\rho-1} F(x)\, dx \right) dy \\
&= \left(\frac{\rho^2\, \xi^2 \big(u_n^{2(1-1/\xi\rho+1/\xi)}\big)}{(\rho\,\xi+\rho-1)^2} - \frac{2\,\rho^2\,\xi^2 \big(u_n^{2-2/\xi\rho+1/\xi}\big)}{(\xi\,\rho-1)\,(2\,\rho\,\xi+\rho-2)} \right) O(1)\ ,
\end{aligned}
$$

we conclude that

$$
\gamma_n = n^{\alpha(\xi-1/\rho)}\, O(1)\ .
$$

Now, we show that

$$
\frac{\sqrt{n}}{\gamma_n} \big(\overline{Z} - \mathbb{E}[Z_1]\big) \xrightarrow{\mathcal{D}} \mathcal{N}(0,1)\,, \qquad \text{as} \quad n \to \infty\ .
$$

With Lindeberg–Feller Theorem (see e.g. Chapter 2 in Durrett (1996)), note that

$$
\begin{aligned}
\frac{\sqrt{n}}{\gamma_n} \big(\overline{Z} - \mathbb{E}[Z_1]\big) &= \sum_{k=1}^{n} \frac{\int_0^{u_n} \big(\overline{F}(x)\big)^{1/\rho-1}\, \mathbf{1}(X_k \le x)\, dx \;-\; \mathbb{E}[Z_1]}{\gamma_n \sqrt{n}} \\
&= \sum_{k=1}^{n} S_{k,n}\ ,
\end{aligned}
$$

where

$$
\mathbb{E}(S_{k,n}) = 0\,, \quad \mathbb{E}(S_{k,n}^2) = 1/n \quad \text{and} \quad \sum_{k=1}^{n} \mathbb{E}(S_{k,n}^2) = 1 \qquad \text{for all} \quad n \ge 1\ .
$$

We need to show that

$$
\sum_{k=1}^{n} \mathbb{E}\Big[|S_{k,n}|^2\, \mathbf{1}\big(|S_{k,n}| > \epsilon\big)\Big] \to 0\,, \qquad \text{as} \quad n \to \infty\ .
$$

Indeed, we have

$$
\sum_{k=1}^{n} \mathbb{E}\Big[|S_{k,n}|^2\, \mathbf{1}\big(|S_{k,n}| > \epsilon\big)\Big] \;=\; \frac{1}{\gamma_n^2}\, \mathbb{E}\Big[\big[Z_k - \mathbb{E}[Z_1]\big]^2\, \mathbf{1}\Big(\big|Z_k - \mathbb{E}[Z_1]\big| > \epsilon\,\gamma_n \sqrt{n}\Big)\Big]\,.
$$

Since $\big|Z_k - \mathbb{E}[Z_1]\big| \le u_n$, then the right side of the previous inequality is less or equal than

$$
\frac{u_n^2}{\gamma_n^2}\, \mathbb{E}\Big[\mathbf{1}\big|Z_k - \mathbb{E}[Z_1]\big| > \epsilon\,\gamma_n \sqrt{n}\Big] \;=\; \frac{u_n^2}{\gamma_n^2}\, \mathbb{P}\Big[\big|Z_k - \mathbb{E}[Z_1]\big| > \epsilon\,\gamma_n \sqrt{n}\Big]\,.
$$

In view of Tchebychev's inequality, we get

$$
\frac{u_n^2}{\gamma_n^2}\, \mathbb{P}\Big[\big|Z_k - \mathbb{E}[Z_1]\big| > \epsilon\,\gamma_n \sqrt{n}\Big] \;\le\; \frac{u_n^2}{\gamma_n^2}\, \frac{1}{\big(\epsilon\,\gamma_n \sqrt{n}\big)^2}\ .
$$

Further, for all $\alpha \in (0,1)$, $\xi \in (0,1)$ and $\epsilon > 0$, with $u_n = O(n^{\alpha\xi})$ was used, then

$$\frac{u_n^2}{\epsilon\, n\, \gamma_n^4} = n^{-2\alpha\xi + 4\alpha/\rho - 1}\, O(1)\ .$$

We must assume: $4\alpha/\rho - 2\,\alpha\xi < 1$ for that $\sum_{k=1}^{n} \mathbb{E}\Big[|S_{k,n}|^2\, \mathbf{1}\big(|S_{k,n}| > \epsilon\big)\Big] \to 0$ as $n \to \infty$.

Finally, we obtain that

$$\begin{aligned}\frac{\sqrt{n}}{\gamma_n}\big(\widehat{\Pi}_{\rho,n} - \Pi_\rho\big) \;\to\; & -\frac{1}{\rho}\frac{\sqrt{n}}{\gamma_n}\big(\overline{Z} - \mathbb{E}[Z_1]\big) + \theta_1 \frac{\sqrt{p_n(1-p_n)}}{\gamma_n}\frac{\sqrt{n}\,(\widehat{p}_n - p_n)}{\sqrt{p_n(1-p_n)}} \\ & + \frac{\theta_2\,\beta_n}{\sqrt{p_n}\,\gamma_n}\sqrt{np_n}\,\big(\widehat{\beta}_n/\beta_n - 1\big) + \frac{\theta_3}{\sqrt{p_n}\,\gamma_n}\sqrt{np_n}\,\big(\widehat{\xi}_n - \xi\big) + o_{\mathbb{P}}(1)\ ,\end{aligned}$$

This enable us to rewrite into

$$\begin{aligned}\frac{\sqrt{n}}{\gamma_n}\big(\widehat{\Pi}_{\rho,n} - \Pi_\rho\big) \;\to\; & -\frac{1}{\rho}\,\mathcal{W}_1 + \theta_1\frac{\sqrt{p_n(1-p_n)}}{\gamma_n}\,\mathcal{W}_2 \\ & + \frac{\sqrt{2\,(1+\xi)}\,\theta_2\,\beta_n}{\sqrt{p_n}\,\gamma_n}\,\mathcal{W}_3 + \frac{(1+\xi)\,\theta_3}{\sqrt{p_n}\,\gamma_n}\,\mathcal{W}_4 + o_{\mathbb{P}}(1)\ ,\end{aligned}$$

where $(\mathcal{W}_i)_{i=1,4}$ are standard normal rv's with $E[\mathcal{W}_i\mathcal{W}_j] = 0$ for every $i,j = 1,...,4$, except for

$$\begin{aligned}E[\mathcal{W}_3\mathcal{W}_4] &= E\left[\frac{1}{\sqrt{2\,(1+\xi)}}\sqrt{np_n}\,\big(\widehat{\beta}_n/\beta_n - 1\big)\,\frac{1}{(1+\xi)}\sqrt{np_n}\,\big(\widehat{\xi}_n - \xi\big)\right] \\ &= \frac{1}{(1+\xi)\,\sqrt{2\,(1+\xi)}}\,E\Big[\sqrt{np_n}\,\big(\widehat{\beta}_n/\beta_n - 1\big)\sqrt{np_n}\,\big(\widehat{\xi}_n - \xi\big)\Big] \\ &= -\frac{1}{\sqrt{2\,(1+\xi)}}\ .\end{aligned}$$

From **Lemma A-2** of Johansson 2003, under the assumptions of Theorem 2.1, we have, for any real numbers, t_1, t_2, t_3 and t_4,

$$\begin{aligned}&\mathbb{E}\left[\exp\left\{i\,t_1\frac{\sqrt{n}}{\gamma_n}\big(\overline{Z} - \mathbb{E}[Z_1]\big) + i\,\sqrt{np_n}\,(t_2, t_3)\begin{pmatrix}\widehat{\beta}_n/\beta - 1\\ \widehat{\xi}_n - \xi\end{pmatrix} + i\,t_4\frac{\sqrt{n}\,(\widehat{p}_n - p_n)}{\sqrt{p_n(1-p_n)}}\right\}\right] \\ &\qquad \to\; \exp\left\{-\frac{t_1^2}{2} - \frac{1}{2}(t_2, t_3)\,\mathbb{Q}^{-1}\begin{pmatrix}t_2\\ t_3\end{pmatrix} - \frac{t_4^2}{2}\right\}\big(1 + o_{\mathbb{P}}(1)\big)\end{aligned}$$

as $n \to \infty$, where $\mathbb{Q}^{-1}$ is that in (2.6), $\gamma_n^2 = \mathrm{Var}(Z_1)$ and $i^2 = -1$.
It follows that, with this result that

$$\frac{\sqrt{n}}{\gamma_n\,\sigma_n}\big(\widehat{\Pi}_{\rho,n} - \Pi_\rho\big) \overset{\mathcal{D}}{\to} \mathcal{N}(0,1)\,, \qquad \text{as} \quad n \to \infty\ ,$$

where

$$\begin{aligned}\sigma_n^2 := &\ \frac{1}{\rho^2} + \frac{\theta_1^2}{\gamma_n^2}\,p_n(1-p_n) + \frac{2\,(1+\xi)\,\theta_2^2\,\beta_n^2}{p_n\gamma_n^2} \\ & + \frac{(1+\xi)^2\,\theta_3^2}{p_n\gamma_n^2} - 2\,\frac{(1+\xi)\,\beta_n\theta_2\,\theta_3}{p_n\gamma_n^2}\ .\end{aligned}$$

This complete the proof of Theorem (2.1). □

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