REVSTAT – Statistical Journal
Volume 11, Number 2, June 2013, 191–210

AN EXPONENTIAL-NEGATIVE BINOMIAL DISTRIBUTION

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Received: July 2012 Revised: August 2012 Accepted: August 2012

Abstract:

- A new three-parameter distribution is proposed for modeling lifetime data. It is advocated as most reasonable among the many exponential mixture type distributions proposed in recent years. An account of the mathematical properties of the new distribution including estimation issues is presented. Two real data applications are described to show superior performance versus at least four of the known lifetime models.

Key-Words:

- *exponential distribution; maximum likelihood estimation; negative binomial distribution.*

AMS Subject Classification:

- Primary: 62E15; Secondary: 62E20.

1. INTRODUCTION

The exponential distribution is the first and most popular model for failure times. In recent years, many authors have proposed generalizations of the exponential distribution. The generalizations are based on a "failure of a system" framework.

Suppose a series system is made up of Z unknown independent components. (The variable Z could be determined by such factors as economy, man power, and customer demand.) Let $Y_1, Y_2, ..., Y_Z$ denote the failure times of the Z components, assumed to be independent of Z. Then the system lifetime is $X = \min(Y_1, Y_2, ..., Y_Z)$. It is reasonable to assume that Y_js are exponential random variables, so the cumulative distribution function (cdf) and the probability density function (pdf) of X are

$$F_X(x) = 1 - \sum_{n=0}^{\infty} \exp(-n\beta x) \Pr(Z = n) \tag{1.1}$$

and

$$f_X(x) = \beta \sum_{n=0}^{\infty} n \exp(-n\beta x) \Pr(Z = n), \tag{1.2}$$

respectively, for $x > 0$ and $\beta > 0$.

Several authors have constructed models for (1.1) and (1.2) by taking Z to follow different distributions. Models with Z belonging to the *Panjer class* (Panjer [15]) have widespread applications in risk theory. The Panjer class includes the geometric, Poisson, negative binomial and other distributions. Panjer [15]'s paper was a breakthrough on the iterative computation of the distribution of aggregate claims, see, for example, Rolski *et al.* [18]. Extended versions of the Panjer class have been introduced by Sundt and Jewell [19], Hess *et al.* [7] and Pestana and Velosa [16]. Panjer class is also used in other contexts, see, for example, Katz [9].

Adamidis and Loukas [1] take Z to be a geometric random variable with parameter p, so yielding

$$f_X(x) = \frac{\beta(1-p)\exp(-\beta x)}{[1 - p\exp(-\beta x)]^2} \tag{1.3}$$

for $x > 0$, $0 < p < 1$ and $\beta > 0$. The case of Z being geometric has been considered much earlier by Rényi [17] in the context of rarefection and by Gnedenko and Korolev [5] and Kovalenko [10] with applications to reliability. We shall refer to (1.3) as the EG distribution. Kus [11] and Hemmati *et al.* [6] take Z to be a Poisson random variable with parameter λ, so yielding

$$f(x) = \frac{\lambda\beta}{1 - \exp(-\lambda)} \exp\{-\lambda - \beta x + \lambda \exp(-\beta x)\} \tag{1.4}$$

for $x > 0$, $\lambda > 0$ and $\beta > 0$. We shall refer to this as the EP distribution. Tahmasbi and Rezaei [20] take Z to be a logarithmic random variable with parameter p, so yielding

$$f(x) = -\frac{1}{\log p}\frac{\beta(1-p)\exp(-\beta x)}{1-(1-p)\exp(-\beta x)} \tag{1.5}$$

for $x > 0$, $0 < p < 1$ and $\beta > 0$. We shall refer to this as the EL distribution. We are not aware of any other model for (1.1) and (1.2) considered in the literature.

In this paper, we propose a new model for (1.1) and (1.2). We take Z to be a negative binomial random variable given by the probability mass function (pmf)

$$f_Z(z) = \binom{z-1}{k-1}(1-p)^k p^{z-k} \tag{1.6}$$

for $z = k, k+1, \ldots$. Geometric pmf is a particular case of (1.6). Poisson pmf is a limiting case of (1.6). Then (1.1) and (1.2) reduce to

$$F_X(x) = 1 - \frac{(1-p)^k\exp(-k\beta x)}{[1-p\exp(-\beta x)]^k} \tag{1.7}$$

and

$$f_X(x) = \frac{k\beta(1-p)^k\exp(-k\beta x)}{[1-p\exp(-\beta x)]^{k+1}}, \tag{1.8}$$

respectively, for $x > 0$, $k > 0$, $0 < p < 1$ and $\beta > 0$. The corresponding hazard rate function (hrf) is

$$h_X(x) = \frac{k\beta}{1-p\exp(-\beta x)} \tag{1.9}$$

for $x > 0$, $k > 0$, $0 < p < 1$ and $\beta > 0$. The corresponding quantile function is

$$F^{-1}(u) = \frac{1}{\beta}\log\left[p + \frac{1-p}{(1-u)^{1/k}}\right] \tag{1.10}$$

for $0 < u < 1$. We shall refer to the distribution given by (1.7) and (1.8) as the exponential negative binomial (ENB) distribution. The exponential distribution arises as the particular case for $k = 1$ and $p = 0$. The EG distribution of Adamidis and Loukas [1] arises as the particular case for $k = 1$.

Note that $d\log f(x)/dx < 0$ for all $x > 0$, so $f(x)$ is a monotonically decreasing function all the time. Note also that $f(0) = k\beta/(1-p)$, $f(\infty) = 0$ and $f(x) \sim k\beta(1-p)^k\exp(-k\beta x)$ as $x \to \infty$. So, the pdf takes a finite value at $x = 0$ and has an exponentially decaying upper tail. Clearly, the hrf given by (1.9) is also a monotonically decreasing function with $h(0) = k\beta/(1-p)$ and $h(\infty) = k\beta$.

Figure 1 illustrates possible shapes of (1.8) for selected parameter values. Figure 2 illustrates possible shapes of (1.9) for selected parameter values. The upper tails of (1.8) become lighter with increasing p and with increasing k. The upper tails of (1.9) become heavier with increasing p and become lighter with increasing k.

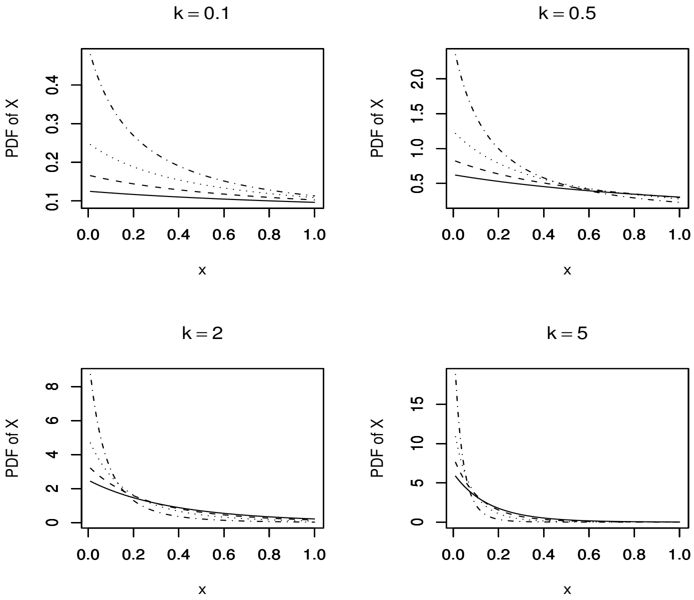

Figure 1: Plots (1.8) versus x for $\beta = 1$, $k = 0.1, 0.5, 2, 5$, $p = 0.2$ (solid curve), $p = 0.4$ (curve of dashes), $p = 0.6$ (curve of dots) and $p = 0.8$ (curve of dots and dashes).

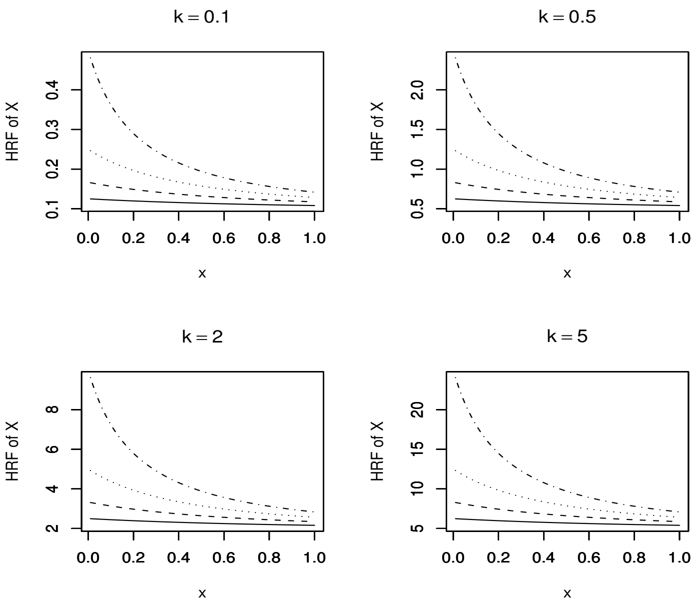

Figure 2: Plots (1.9) versus x for $\beta = 1$, $k = 0.1, 0.5, 2, 5$, $p = 0.2$ (solid curve), $p = 0.4$ (curve of dashes), $p = 0.6$ (curve of dots) and $p = 0.8$ (curve of dots and dashes).

The new distribution given by (1.7) and (1.8) can be motivated in several different ways. Firstly, the negative binomial distribution is a generalization of the geometric and Poisson distributions (Poisson is a limiting particular case). The negative binomial distribution with support over the set of all non-negative integers is also a generalization of the Poisson distribution in the sense that it can deduced as a hierarchical model if $X \sim$ Poisson (Λ) with Λ being a gamma random variable, see, for example, Casella and Berger [3].

So, (1.8) can be considered a generalization of (1.3) and (1.4). The logarithmic distribution is used to construct (1.5). The logarithmic distribution is widely used in population studies, iteration, fractality and chaos. But it is not a well known model for counts as the geometric, Poisson and negative binomial distributions are.

Secondly, using the series expansion

$$(1-a)^{-k-1} = \sum_{i=0}^{\infty} \binom{-k-1}{i} (-a)^i,$$

we can rewrite (1.8) as

$$f_X(x) = k\beta(1-p)^k \sum_{i=0}^{\infty} \binom{-k-1}{i} (-p)^i \exp\left\{-(k+i)\beta x\right\}. \tag{1.11}$$

Integrating (1.11), we can rewrite (1.7) as

$$F_X(x) = 1 - k(1-p)^k \sum_{i=0}^{\infty} \binom{-k-1}{i} \frac{(-p)^i}{k+i} \exp\left\{-(k+i)\beta x\right\}. \tag{1.12}$$

It follows from (1.11) and (1.12) that the ENB distribution is a mixture of the exponential distribution, the earliest and the best known model for failure times.

Our third motivation is simulation based. We shall see later (see Section 6) that the ENB distribution provides significantly better fits than the EG, EP and EL distributions, the only known competing distributions under the framework of (1.1) and (1.2), for more than tens of thousands of simulated samples. This is the case even when the samples are simulated from the EG, EP and EL distributions.

Our fourth and final motivation is real data based. We shall see later (see Section 7) that the proposed distribution outperforms the EP and EL distributions as well as the two-parameter Weibull distribution and the three-parameter Weibull Poisson distribution (Hemmati *et al.* [6]) with respect to at least two real data sets.

The contents of this paper are organized as follows. An account of mathematical properties of the new distribution is provided in Sections 2 to 4. The properties studied include: raw moments, order statistics and their moments, and asymptotic distribution of the extreme values. Estimation by the methods of moments and maximum likelihood is presented in Section 5. A simulation study to compare the performance of the proposed distribution versus the EG, EP and EL distributions is presented in Section 6. Finally, Section 7 illustrates an application by using two real data sets.

2. MOMENTS

Let X denote a random variable with the pdf (1.8). It follows from Lemma A.1 in the Appendix that

$$\begin{aligned} E\left(X^n\right) &= \frac{n!(1-p)^k}{\beta^n k^n} {}_{n+2}F_{n+1}\left(1+k, k, ..., k; k+1, ..., k+1; p\right) \\ &= \frac{n!(1-p)^k}{\beta^n k^n} {}_{n+1}F_{n}\left(k, ..., k; k+1, ..., k+1; p\right), \end{aligned}$$

where ${}_pF_q(a_1, a_2, ..., a_p; b_1, b_2, ..., b_q; x)$ denotes the generalized hypergeometric function defined by

$${}_pF_q\left(a_1, a_2, ..., a_p; b_1, b_2, ..., b_q; x\right) = \sum_{k=0}^{\infty} \frac{\left(a_1\right)_k \left(a_2\right)_k \cdots \left(a_p\right)_k}{\left(b_1\right)_k \left(b_2\right)_k \cdots \left(b_q\right)_k} \frac{x^k}{k!},$$

where $(e)_k = e(e+1)\cdots(e+k-1)$ denotes the ascending factorial. In particular, the first four moments of X are

$$E\left(X\right) = \frac{(1-p)^k}{\beta k} {}_2F_1\left(k, k; k+1; p\right),$$

$$E\left(X^2\right) = \frac{2(1-p)^k}{\beta^2 k^2} {}_3F_2\left(k, k, k; k+1, k+1; p\right),$$

$$E\left(X^3\right) = \frac{6(1-p)^k}{\beta^3 k^3} {}_4F_3\left(k, k, k, k; k+1, k+1, k+1; p\right)$$

and

$$E\left(X^4\right) = \frac{24(1-p)^k}{\beta^4 k^4} {}_5F_4\left(k, k, k, k, k; k+1, k+1, k+1, k+1; p\right).$$

The variance, skewness and kurtosis of X can be obtained using the relationships $Var(X) = E(X^2) - (E(X))^2$, $Skewness(X) = E(X - E(X))^3/(Var(X))^{3/2}$ and $Kurtosis(X) = E(X - E(X))^4/(Var(X))^2$. The variations of E (X), Var (X), Skewness (X) and Kurtosis (X) versus k and p for $\beta = 1$ are illustrated in Figure 3. It appears that $E(X)$ and $Var(X)$ are decreasing functions with respect to both k and p. Skewness (X) and Kurtosis (X) appear to increase with respect to a. With respect to p, they initially increase before decreasing.

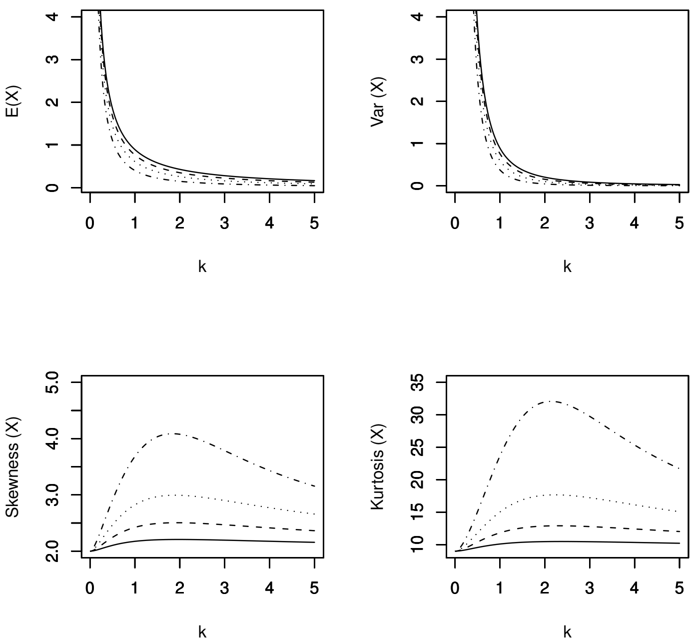

Figure 3: Mean, variance, skewness and kurtosis for (1.8) versus k for $p = 0.2$ (solid curve), $p = 0.4$ (curve of dashes), $p = 0.6$ (curve of dots) and $p = 0.8$ (curve of dots and dashes).

3. ORDER STATISTICS

Suppose $X_1, X_2, ..., X_n$ is a random sample from (1.8). Let $X_{1:n} < X_{2:n} < \cdots < X_{n:n}$ denote the corresponding order statistics. It is well known that the pdf and the cdf of the rth order statistic, say $Y = X_{r:n}$, are given by

$$\begin{aligned} f_Y(y) &= \frac{n!}{(r-1)!(n-r)!} F_X^{r-1}(y)\left\{1 - F_X(y)\right\}^{n-r} f_X(y) \\ &= \frac{n!}{(r-1)!(n-r)!} \sum_{\ell=0}^{n-r} \binom{n-r}{\ell} (-1)^\ell F_X^{r-1+\ell}(y) f_X(y) \end{aligned}$$

and

$$F_Y(y) = \sum_{j=r}^{n} \binom{n}{j} F_X^j(y)\left\{1 - F_X(y)\right\}^{n-j} = \sum_{j=r}^{n} \sum_{\ell=0}^{n-j} \binom{n}{j} \binom{n-j}{\ell} (-1)^\ell F_X^{j+\ell}(y),$$

respectively, for $r = 1, 2, ..., n$. It follows from (1.8) and (1.7) that

$$f_Y(y) = \frac{k\beta n!}{(r-1)!(n-r)!} \sum_{\ell=0}^{n-r} \binom{n-r}{\ell} (-1)^\ell \frac{(1-p)^{(r+\ell)k} \exp\left[-(r+\ell)k\beta y\right]}{\left[1 - p\exp(-\beta y)\right]^{(r+\ell)k+1}}$$

and

$$F_Y(y) = \sum_{j=r}^{n} \sum_{\ell=0}^{n-j} \binom{n}{j} \binom{n-j}{\ell} (-1)^{\ell} \frac{(1-p)^{(j+\ell)k} \exp\left[-(j+\ell)k\beta y\right]}{\left[1 - p\exp(-\beta y)\right]^{(j+\ell)k}}.$$

Using Lemma A.1 in the Appendix, the qth moment of Y can be expressed as

$$E\left(Y^q\right) = \frac{q!n!}{\beta^q k^q (r-1)!(n-r)!} \sum_{\ell=0}^{n-r} \binom{n-r}{\ell} \frac{(-1)^{\ell}(1-p)^{(r+\ell)k}}{(r+\ell)^{q+1}} G(\ell)$$

for $q \geq 1$, where $G(\ell) = {}_{q+1}F_q(\ell k + rk, ..., \ell k + rk; 1 + \ell k + rk, ..., 1 + \ell k + rk; p)$.

4. EXTREME VALUES

If $\overline{X} = (X_1 + \cdots + X_n)/n$ denotes the sample mean then by the usual central limit theorem $\sqrt{n}(\overline{X} - E(X))/\sqrt{Var(X)}$ approaches the standard normal distribution as $n \to \infty$. Sometimes one would be interested in the asymptotics of the extreme values $M_n = \max(X_1, ..., X_n)$ and $m_n = \min(X_1, ..., X_n)$.

Let $g(t) = 1/(k\beta)$. Take the cdf and the pdf as specified by (1.7) and (1.8), respectively. Since $f(x) \sim k\beta(1-p)^k \exp(-k\beta x)$ as $x \to \infty$,

$$\lim_{t \to \infty} \frac{1 - F(t + xg(t))}{1 - F(t)} = \lim_{t \to \infty} \frac{f(t + x/(k\beta))}{f(t)} = \exp(-x).$$

Since $f(0) = k\beta/(1-p)$,

$$\lim_{t \to 0} \frac{F(tx)}{F(t)} = \lim_{t \to \infty} \frac{xf(tx)}{f(t)} = x.$$

Hence, it follows from Theorem 1.6.2 in Leadbetter *et al.* [12] that there must be norming constants $a_n > 0$, b_n, $c_n > 0$ and d_n such that

$$\Pr\left\{a_n\left(M_n - b_n\right) \leq x\right\} \to \exp\left\{-\exp(-x)\right\}$$

and

$$\Pr\left\{c_n\left(m_n - d_n\right) \leq x\right\} \to 1 - \exp\left(-x\right)$$

as $n \to \infty$. The form of the norming constants can also be determined. For instance, using Corollary 1.6.3 in Leadbetter *et al.* [12], one can see that $b_n = F^{-1}(1 - 1/n)$ and $a_n = k\beta$, where $F^{-1}(\cdot)$ is given by (1.10).

5. ESTIMATION

Here, we consider estimation by the methods of moments and maximum likelihood and provide expressions for the associated Fisher information matrix.

Suppose $x_1, ..., x_n$ is a random sample from (1.8). For moments estimation, let $m_1 = (1/n)\sum_{j=1}^{n} x_j$, $m_2 = (1/n)\sum_{j=1}^{n} x_j^2$ and $m_3 = (1/n)\sum_{j=1}^{n} x_j^3$. By equating the theoretical moments of (1.8) with the sample moments, we obtain the equations:

$$\frac{(1-p)^k}{\beta k} {}_2F_1\left(k, k; k+1; p\right) = m_1,$$

$$\frac{2(1-p)^k}{\beta^2 k^2} {}_3F_2\left(k, k, k; k+1, k+1; p\right) = m_2,$$

and

$$\frac{6(1-p)^k}{\beta^3 k^3} {}_4F_3\left(k, k, k, k; k+1, k+1, k+1; p\right) = m_3.$$

The method of moments estimators (mmes), say $\widetilde{p}$, $\widetilde{k}$ and $\widetilde{\beta}$, are the simultaneous solutions of these three equations.

Now consider estimation by the method of maximum likelihood. The log likelihood function of the three parameters is:

$$\begin{aligned} \log L(p, k, \beta) &= n \log(k\beta) + nk \log(1-p) - k\beta \sum_{i=1}^{n} x_i \\ &\quad -(k+1) \sum_{i=1}^{n} \log\left[1 - p \exp\left(-\beta x_i\right)\right]. \end{aligned} \tag{5.1}$$

It follows that the maximum likelihood estimators (mles), say $\widehat{p}$, $\widehat{k}$ and $\widehat{\beta}$, are the simultaneous solutions of the equations:

$$\frac{n}{k} + n \log(1-p) = \beta \sum_{i=1}^{n} x_i + \sum_{i=1}^{n} \log\left[1 - p \exp\left(-\beta x_i\right)\right],$$

$$\frac{n}{\beta} = k \sum_{i=1}^{n} x_i + p(k+1) \sum_{i=1}^{n} \frac{x_i \exp\left(-\beta x_i\right)}{1 - p \exp\left(-\beta x_i\right)},$$

and

$$\frac{nk}{1-p} = (k+1) \sum_{i=1}^{n} \frac{\exp\left(-\beta x_i\right)}{1 - p \exp\left(-\beta x_i\right)}.$$

For interval estimation of (p, k, β) and tests of hypothesis, one requires the Fisher information matrix:

$$\mathbf{I} = \begin{pmatrix} E\left(-\frac{\partial^2 \log L}{\partial p^2}\right) & E\left(-\frac{\partial^2 \log L}{\partial p \partial k}\right) & E\left(-\frac{\partial^2 \log L}{\partial p \partial \beta}\right) \\ E\left(-\frac{\partial^2 \log L}{\partial k \partial p}\right) & E\left(-\frac{\partial^2 \log L}{\partial k^2}\right) & E\left(-\frac{\partial^2 \log L}{\partial k \partial \beta}\right) \\ E\left(-\frac{\partial^2 \log L}{\partial \beta \partial p}\right) & E\left(-\frac{\partial^2 \log L}{\partial \beta \partial k}\right) & E\left(-\frac{\partial^2 \log L}{\partial \beta^2}\right) \end{pmatrix}.$$

Using Lemma A.1 in the Appendix, the elements of this matrix for (5.1) can be worked out as:

$$I_{11} = \frac{n}{k^2},$$

$$\begin{aligned} I_{12} = I_{21} &= \frac{n(1-p)^k}{\beta k} {}_2F_1\left(k, k; k+1; p\right) \\ &+ \frac{npk(1-p)^k}{\beta (k+1)^2} {}_3F_2\left(k+3, k+1, k+1; k+2, k+2; p\right), \end{aligned}$$

$$I_{13} = I_{31} = \frac{n}{1-p} - \frac{nk}{(k+1)(1-p)},$$

$$I_{22} = \frac{n}{\beta^2} - \frac{npk(1-p)^k}{\beta^2 (k+1)^2} {}_3F_2\left(k+3, k+1, k+1; k+2, k+2; p\right),$$

$$\begin{aligned} I_{23} = I_{32} &= \frac{nk(1-p)^k}{\beta (k+1)} {}_3F_2\left(k+3, k+1, k+1; k+2, k+2; p\right) \\ &+ \frac{npk(k+1)(1-p)^k}{\beta (k+2)^2} {}_2F_1\left(k+2, k+2; k+3; p\right), \end{aligned}$$

and

$$I_{33} = \frac{nk}{(1-p)^2} - \frac{nk(k+1)}{(k+2)(1-p)^2}.$$

Under regularity conditions, the asymptotic distribution of $(\widehat{p}, \widehat{k}, \widehat{\beta})$ as $n \to \infty$ is trivariate normal with zero means and variance co-variance matrix $\mathbf{I}^{-1}$. So, $\mathrm{Var}\,(\widehat{p}) = (I_{33}I_{22} - I_{32}I_{23})/\Delta$, $\mathrm{Cov}\,(\widehat{p}, \widehat{k}) = -(I_{33}I_{12} - I_{32}I_{13})/\Delta$, $\mathrm{Cov}\,(\widehat{p}, \widehat{\beta}) = (I_{23}I_{12} - I_{22}I_{13})/\Delta$, $\mathrm{Var}\,(\widehat{k}) = (I_{33}I_{11} - I_{31}I_{13})/\Delta$, $\mathrm{Cov}\,(\widehat{k}, \widehat{\beta}) = -(I_{23}I_{11} - I_{21}I_{13})/\Delta$ and $\mathrm{Var}\,(\widehat{\beta}) = (I_{22}I_{11} - I_{21}I_{12})/\Delta$, where $\Delta = I_{11}(I_{33}I_{22} - I_{32}I_{23}) - I_{21}(I_{33}I_{12} - I_{32}I_{13}) + I_{31}(I_{23}I_{12} - I_{22}I_{13})$.

6. A SIMULATION STUDY

Here, we perform a simulation study to compare the performance of the proposed distribution versus those given by (1.3), (1.4) and (1.5); that is, the EG, EP and EL distributions, the only known competing distributions under the framework of (1.1) and (1.2). We use the following scheme:

1. Generate ten thousand samples of size n from (1.8);

2. For each sample, fit the models given by (1.8), (1.3), (1.4) and (1.5);

3. Let ℓ_{1i}, ℓ_{2i}, ℓ_{3i} and ℓ_{4i}, $i = 1, 2, ..., 10000$ denote the maximized log-likelihoods for (1.8), (1.3), (1.4) and (1.5) for the ten thousand samples;

4. Draw the box plots of $2(\ell_{1i} - \ell_{2i})$, $2(\ell_{1i} - \ell_{3i})$ and $2(\ell_{1i} - \ell_{4i})$, $i = 1, 2, ..., 10000$.

This scheme compares the fits of the four distributions when simulated samples are from the proposed distribution. For completeness, we repeated the above scheme with simulated samples coming from the EG, EP and EL distributions.

The resulting box plots are shown in Figure 4 for $n = 25$ and $(\beta, \lambda, k, p) = (1, 1, 2, 0.5)$. The figure shows that proposed distribution provides the best fit wherever the sample comes from. The relative performances of the EG, EP and EL distributions with respect to the proposed one appear similar. The four distributions are not nested, so the likelihood ratio test may not be used to discriminate between them. But the differences in the log-likelihood are so large that they are significant even with respect to the AIC and BIC criteria.

For reasons of space, we have presented results for only one value for n and the parameters. But the conclusions of Figure 4 hold also for larger sample sizes and other parameter values.

The results are not surprising because, as explained in Section 1, the proposed distribution is flexible enough to contain the EG and EP distributions as particular cases. The logarithmic distribution used to construct the EL distribution is not flexible and is certainly not widely used.

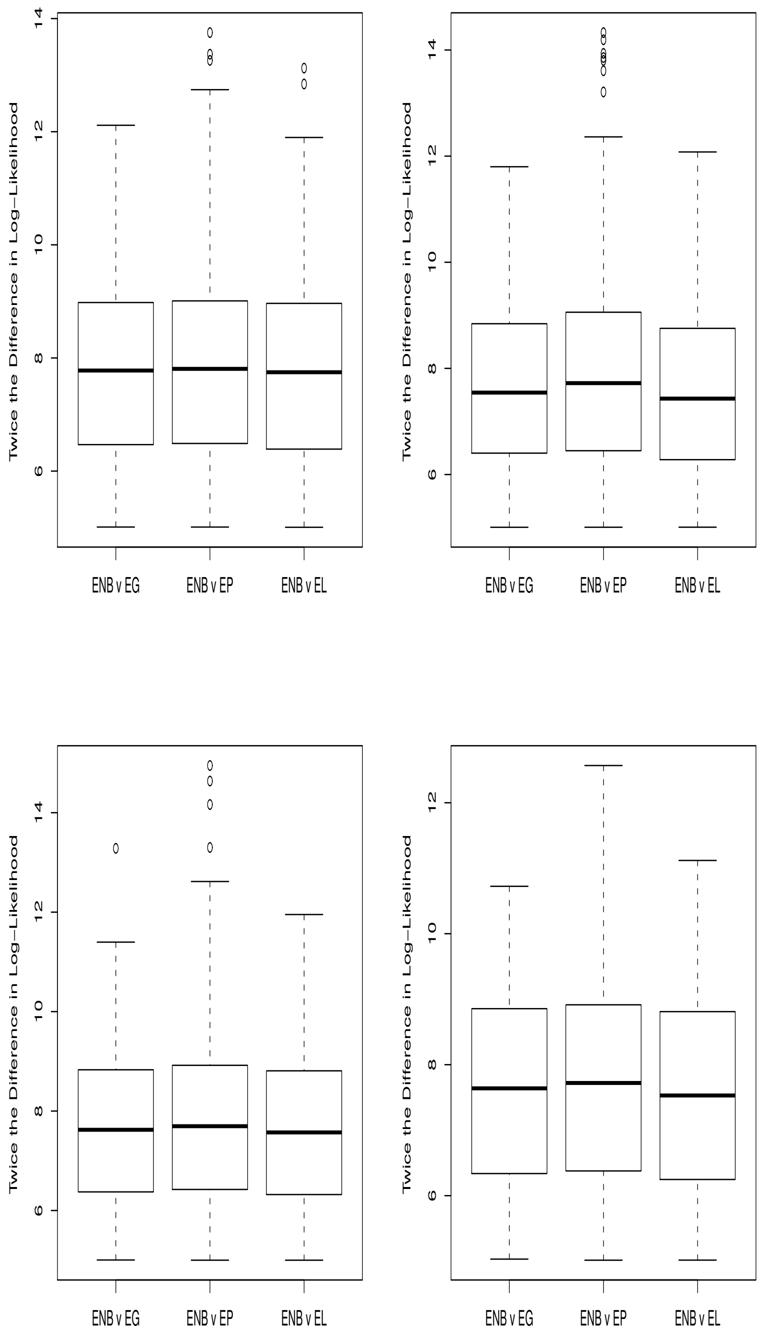

Figure 4: Box plots of $2(\ell_{1i} - \ell_{2i})$, $2(\ell_{1i} - \ell_{3i})$ and $2(\ell_{1i} - \ell_{4i})$ when the simulated samples are from the proposed distribution (top left), the EG distribution (top right), the EP distribution (bottom left) and the EL distribution (bottom right).

7. APPLICATIONS

Here, we illustrate applicability of the ENB distribution using two real data sets. The first data set contains intervals in days between successive failures of a piece of software. See Jelinski and Moranda [8] and Linda [13]. The second data set consists of lifetimes of pressure vessels. See Pal *et al.* [14].

We compare the fit of the ENB distribution with those of the EP and EL distributions as well as those of the Weibull distribution given by the pdf

$$f(x) = \beta \lambda^{\beta} x^{\beta-1} \exp\left\{-(\lambda x)^{\beta}\right\}$$

(for $x > 0$, $\lambda > 0$ and $\beta > 0$) and, the Weibull Poisson distribution (Hemmati *et al.* [6]) given by the pdf

$$f(x) = \frac{\theta \alpha \beta^{\alpha}}{\exp(\theta) - 1} \exp\left\{-(\beta x)^{\alpha}\right\} \exp\left\{-\theta \exp\left[-(\beta x)^{\alpha}\right]\right\}$$

for $x > 0$, $\theta > 0$, $\alpha > 0$ and $\beta > 0$. The parameters of the ENB distribution are estimated by the method of maximum likelihood, see Section 5. The parameters of other distributions are also estimated by the method of maximum likelihood.

The mles and the corresponding log-likelihood value, the Kolmogorov Smirnov statistic, its p value, the AIC value and the BIC value are shown in Tables 1 and 2. We can see that the largest log-likelihood value, the largest p value, the smallest AIC value and the smallest BIC value are obtained for the ENB distribution.

Table 1: Fitted estimates for data set 1.

Model	Parameter estimates	Log likelihood	K-S statistic	p-value	AIC	BIC
Weibull	(16.7835, 0.6460)	-131.6366	0.2046	0.1092	267.2732	270.2662
EL	(0.0300, 0.0162)	-129.6636	0.2147	0.0818	263.3273	266.3203
EP	(0.0191, 3.9168)	-131.2939	0.1967	0.1358	266.5878	269.5808
WP	(0.0182, 0.8072, 3.3587)	-129.5968	0.1634	0.3070	265.1936	269.6831
ENB	(0.0076, 0.9491, 0.9462)	-127.7312	0.1372	0.5189	261.4624	265.9519

Table 2: Fitted estimates for data set 2.

Model	Parameter estimates	Log likelihood	K-S statistic	p-value	AIC	BIC
Weibull	(488.1066, 0.7162)	-145.3353	0.1519	0.6904	294.6705	296.6620
EL	(0.1239, 0.0011)	-146.5781	0.1700	0.5531	297.1562	299.1477
EP	(0.0015, 0.6978)	-146.9594	0.1534	0.6787	297.9189	299.9104
WP	(0.0020, 0.7162, 0.0001)	-145.3353	0.1519	0.6904	296.6705	299.6577
ENB	(0.0342, 0.0434, 0.9748)	-143.4332	0.1309	0.8400	292.8665	295.8537

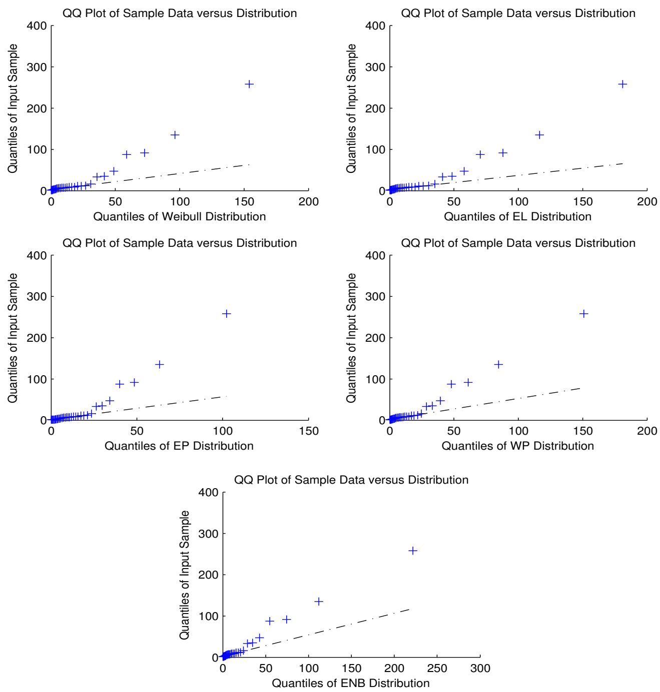

Figure 5: Quantile-quantile plots for the fitted models for the first data set.

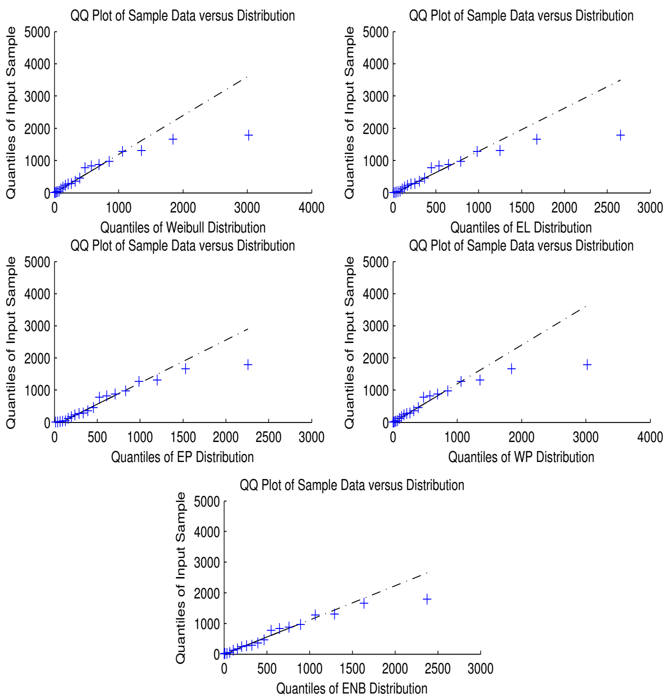

Figure 6: Quantile-quantile plots for the fitted models for the second data set.

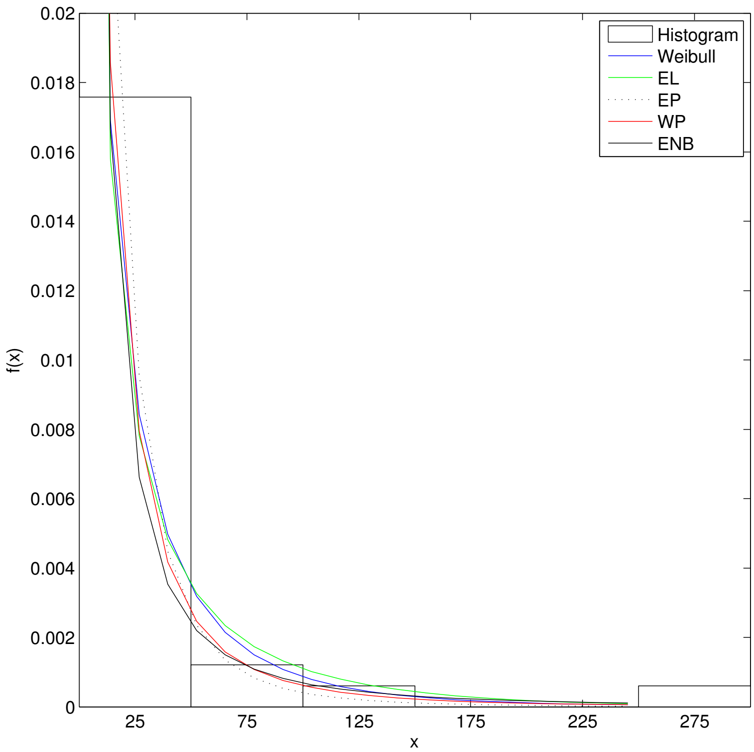

Figure 7: Fitted pdfs and the observed histogram for the first data set.

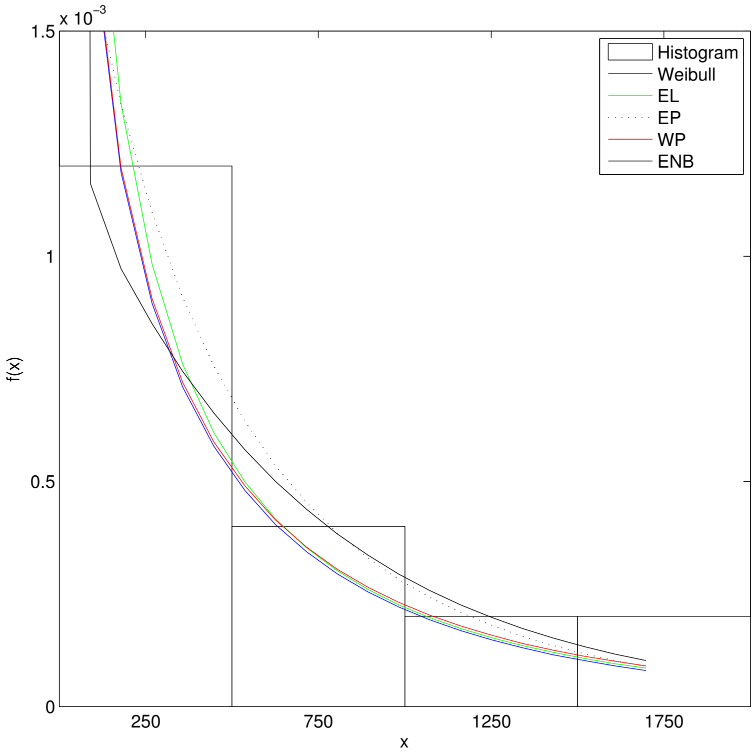

Figure 8: Fitted pdfs and the observed histogram for the second data set.

The conclusion based on Tables 1 and 2 can be verified by means of quantile-quantile plots and density plots. A quantile-quantile plot consists of plots of the observed quantiles against quantiles predicted by the fitted model. For example, for the model based on the ENB distribution, $x_{(j)}$ was plotted versus $F^{-1}((j - 0.375)/(n + 0.25))$, $j = 1, 2, ..., n$, as recommended by Blom [2] and Chambers *et al.* [4], where $F^{-1}(\cdot)$ is given by (1.10), $x_{(j)}$ are the sorted values of the observed data in the ascending order and n is the number of observations. The quantile-quantile plots for the five fitted models and for each data set are shown in Figures 5 and 6. We can see that the model based on the ENB distribution has the points closer to the diagonal line for each data set.

A density plot compares the fitted pdfs of the models with the empirical histogram of the observed data. The density plots for the two data sets are shown in Figures 7 and 8. Again the fitted pdfs for the ENB distribution appear to capture the general pattern of the empirical histograms better.

APPENDIX

We need the following lemma.

Lemma A.1. *Let*

$$I(a,b,c) = k\beta(1-p)^k \int_0^\infty \frac{x^a \exp\left[-(k+b)\beta x\right]}{\left[1-p\,\exp(-\beta x)\right]^{k+1+c}} dx.$$

Then

$$I(a,b,c) = \frac{a!k(1-p)^k {}_{a+2}F_{a+1}\left(1+k+c, k+b, ..., k+b;\; k+b+1, ..., k+b+1;\; p\right)}{\beta^a (k+b)^{a+1}}.$$

Proof: Using the series expansion

$$(1-a)^{-k-1-c} = \sum_{i=0}^{\infty} \binom{-k-1-c}{i} (-a)^i,$$

we can write

$$\begin{aligned} I(a,b,c) &= k\beta(1-p)^k \sum_{i=0}^{\infty} \binom{-k-1-c}{i} (-p)^i \int_0^\infty x^a \exp\left[-(k+b+i)\beta x)\right] dx \\ &= a!k\beta^{-a}(1-p)^k \sum_{i=0}^{\infty} \binom{-k-1-c}{i} \frac{(-p)^i}{(k+b+i)^{a+1}} \\ &= a!k\beta^{-a}(1-p)^k (k+b)^{-a-1} \sum_{i=0}^{\infty} \frac{(k+1+c)_i (k+b)_i \cdots (k+b)_i}{(k+b+1)_i \cdots (k+b+1)_i} \frac{p^i}{i!}. \end{aligned}$$

The result now follows from the definition of hypergeometric functions. □

ACKNOWLEDGMENTS

The authors would like to thank the Editor, the Associate Editor, and the referee for careful reading and for comments which greatly improved the paper.

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