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AN ACCURATE APPROXIMATION TO THE DISTRIBUTION OF A LINEAR COMBINATION OF NON-CENTRAL CHI-SQUARE RANDOM VARIABLES

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Abstract:

- This paper provides an accessible methodology for approximating the distribution of a general linear combination of non-central chi-square random variables. Attention is focused on the main application of the results, namely the distribution of positive definite and indefinite quadratic forms in normal random variables. After explaining that the moments of a quadratic form can be determined from its cumulants by means of a recursive formula, we propose a moment-based approximation of the density function of a positive definite quadratic form, which consists of a gamma density function that is adjusted by a linear combination of Laguerre polynomials or, equivalently, by a single polynomial. On expressing an indefinite quadratic form as the difference of two positive definite quadratic forms, explicit representations of approximations to its density and distribution functions are obtained in terms of confluent hypergeometric functions. The proposed closed form expressions converge rapidly and provide accurate approximations over the entire support of the distribution. Additionally, bounds are derived for the integrated squared and absolute truncation errors. An easily implementable algorithm is provided and several illustrative numerical examples are presented. In particular, the methodology is applied to the Durbin–Watson statistic. Finally, relevant computational considerations are discussed. Linear combinations of chi-square random variables and quadratic forms in normal variables being ubiquitous in statistics, the distribution approximation technique introduced herewith should prove widely applicable.

Key-Words:

- *chi-square random variables; linear combinations; quadratic forms; cumulants; moments; density approximation; Durbin–Watson statistic.*

AMS Subject Classification:

- 62E17, 62Q05.

1. INTRODUCTION

The distribution of linear combinations of chi-square random variables and that of quadratic forms in normal vectors have already received a lot of attention in the statistical literature. Box (1954) considered a linear combination of chi-square variables having even degrees of freedom. Some representations of the density function of linear combinations of chi-square variables were derived by Mathai and Saxena (1978). Various representations of the distribution function of a quadratic form are available, and several procedures have been proposed for computing percentage points and preparing tables. Gurland (1948, 1953, 1956), Pachares (1955), Ruben (1960, 1962), Shah and Khatri (1961), and Kotz *et al.* (1967a,b) among others, have given representations of the distribution function of quadratic forms in terms of MacLaurin series and the distribution function of chi-square variables. Gurland (1956) and Shah (1963) considered respectively central and non-central indefinite quadratic forms, but as pointed by Shah (1963), the expansions obtained are not practical. Press (1967) provided infinite series representations of the density and distribution functions of an indefinite quadratic form in normal variables. Other representations of the exact density and distribution functions of indefinite quadratic forms have been given by Imhof (1961), Davis (1973) and Rice (1980). As pointed out in Mathai and Provost (1992), a wide array of statistics can be expressed in terms of quadratic forms in normal random vectors. For example, one may consider the lagged regression residuals developed by De Gooijer and MacNeill (1999) and discussed in Provost *et al.* (2005), or certain change point test statistics derived by MacNeill (1978). Hillier (2001) expressed the density function of a ratio of quadratic forms in normal random variables in terms of top-order zonal polynomials involving difference quotients of the characteristic roots of the matrix in the numerator quadratic form. The sample serial correlation coefficient as defined in Anderson (1990) and discussed in Provost and Rudiuk (1995) as well as the sample innovation cross-correlation function for an ARMA time series whose asymptotic distribution was derived by McLeod (1979) have such a structure.

Monte Carlo simulations, whereby artificial data are generated and sampling distributions and moments then are estimated, can be implemented more easily on an extensive array of models. These simulations may, however, result in some limitations such as sampling variations and simulation inadequacies, and their results may be specific to the set of parameter values assumed in the simulations. Hendry and Harrison (1974), Dempster *et al.* (1977), Hendry (1979), and Hendry and Mizon (1980) among others, have attempted to cope with these issues. On the other hand, the analytical approach derives results which hold over the entire parameter space but may find some limitations in terms of simplifications on the model, which are imposed to render the problem tractable. The analytical approach has been applied to various statistics involving quadratic forms. Examples in this area include certain heteroscedastic models studied by Taylor (1977, 1978), the first-order autoregressive process considered by Sawa (1978) and Phillips (1977, 1978), the regression models analyzed by Dwivedi and Srivastava (1979), a linear model with unknown covariance structure studied by Yamamoto (1979), as well as the Bayesian analysis of simultaneous equations models carried out by Zellner (1971) and Dreze (1976).

A novel and accessible moment-based approach is proposed in this paper for approximating the density function of positive definite quadratic forms in normal random variables in terms of a gamma density function and a linear combination of Laguerre polynomials, which is re-expressed as a single polynomial so that analytic expressions could also be worked out for the case of indefinite quadratic forms. The resulting closed form density and distribution functions converge rapidly and provide accurate approximations over the entire support of the distribution.

Existing expansions that are expressed in terms of rescaled chi-square density functions and Laguerre polynomials such as those discussed in Kotz *et al.* (1967a,b) for the case of positive definite quadratic forms, were derived by making use of a different technique. As in the case of Edgeworth-type expansions whose leading terms are Gaussian density or distribution functions, such representations cannot converge as quickly as the proposed expansion, which is more appropriately based on a gamma density function whose first two moments match those of the target distribution. It should also be pointed out that the saddlepoint approximation and Imhof's formula, which incidentally is not closed form, need to be recalculated at each point of the distribution. Moreover, as can be seen for instance from Huzurbazar (1999), Figure 2, the saddlepoint approximation may not be accurate throughout the entire range of the distribution.

As will be explained, the results also apply to ratios of certain quadratic forms. Such ratios arise for example in regression theory, linear models, analysis of variance and time series.

A representation of non-central indefinite quadratic forms, which relies on the spectral decomposition theorem, is derived in Section 2; a formula for determining their moments in terms of their cumulants is provided as well. A so-called Laguerre polynomial approximation of the density function of a positive definite quadratic form, which is expressed as the product of a gamma density function and a single polynomial, is introduced in Section 3; explicit representations of the resulting density and distribution functions of an indefinite quadratic form are also given. We note that the expansions are expressed in terms of Laguerre polynomials (or their coefficients) since their associated weight functions are proportional to gamma density functions, which are suitable for approximating the distribution of positive linear combinations of chi-square random variables. An algorithm describing the methodology is provided in Section 4.

Several numerical examples, including an application of the proposed technique to the Durbin–Watson statistic, are presented in Section 5. Finally, certain computational considerations are discussed in Section 6.

2. THE MOMENTS OF A LINEAR COMBINATION OF CHI-SQUARE RANDOM VARIABLES

Since linear combinations of possibly non-central chi-square random variables can be expressed in terms of quadratic forms, we shall provide a representation of the moments of the latter in this section. These moments are required in order to implement the proposed density approximation methodology.

Indefinite quadratic forms in normal random variables can be expressed in terms of standard normal variables as follows. Let $\mathbf{X} \sim \mathcal{N}_p(\boldsymbol{\mu}, \Sigma)$ where Σ is a positive definite covariance matrix. On letting $\mathbf{Z} \sim \mathcal{N}_p(\mathbf{0}, I)$, where I is a $p \times p$ identity matrix, one has $\mathbf{X} = \Sigma^{\frac{1}{2}}\mathbf{Z} + \boldsymbol{\mu}$ where $\Sigma^{\frac{1}{2}}$ denotes the symmetric square root of Σ. Then, the quadratic form $Q = \mathbf{X}'A\mathbf{X}$ where A is a $p \times p$ real symmetric matrix and $\mathbf{X}'$ denotes the transpose of $\mathbf{X}$ can be expressed as follows:

$$\begin{aligned}Q &= \left(\mathbf{Z} + \Sigma^{-\frac{1}{2}}\boldsymbol{\mu}\right)' \Sigma^{\frac{1}{2}} A\, \Sigma^{\frac{1}{2}} \left(\mathbf{Z} + \Sigma^{-\frac{1}{2}}\boldsymbol{\mu}\right) \\ &= \left(\mathbf{Z} + \Sigma^{-\frac{1}{2}}\boldsymbol{\mu}\right)' P P'\, \Sigma^{\frac{1}{2}} A\, \Sigma^{\frac{1}{2}}\, P P' \left(\mathbf{Z} + \Sigma^{-\frac{1}{2}}\boldsymbol{\mu}\right)\end{aligned} \tag{2.1}$$

where P is an orthogonal matrix that diagonalizes $\Sigma^{\frac{1}{2}} A \Sigma^{\frac{1}{2}}$, that is, $P'\Sigma^{\frac{1}{2}} A \Sigma^{\frac{1}{2}} P = \text{diag}(\lambda_1, ..., \lambda_p)$, $\lambda_1, ..., \lambda_p$ being the eigenvalues of $\Sigma^{\frac{1}{2}} A \Sigma^{\frac{1}{2}}$ (or equivalently those of $A\Sigma$) in *decreasing* order. Let $\mathbf{v}_i$ denote the *normalized* eigenvector of $\Sigma^{\frac{1}{2}} A \Sigma^{\frac{1}{2}}$ *corresponding to* λ_i (such that $\Sigma^{\frac{1}{2}} A \Sigma^{\frac{1}{2}} \mathbf{v}_i = \lambda_i \mathbf{v}_i$ *and* $\mathbf{v}_i{}'\mathbf{v}_i = 1$), $i = 1, ..., p$, and $P = (\mathbf{v}_1, ..., \mathbf{v}_p)$. Letting $\mathbf{U} = P'\mathbf{Z}$, one has $\mathbf{U} \sim \mathcal{N}_p(\mathbf{0}, I)$ since P is an orthogonal matrix, and then, according to the spectral decomposition theorem,

$$\begin{aligned}Q &= (\mathbf{U} + \mathbf{b})' \,\text{diag}(\lambda_1, ..., \lambda_p)\, (\mathbf{U} + \mathbf{b}) \\ &= \sum_{j=1}^{p} \lambda_j (U_j + b_j)^2\end{aligned} \tag{2.2}$$

where $\text{diag}(\lambda_1, ..., \lambda_p)$ is a diagonal matrix whose diagonal elements are $\lambda_1, ..., \lambda_p$, $\mathbf{b} = P'\Sigma^{-\frac{1}{2}}\boldsymbol{\mu}$ with $\mathbf{b} = (b_1, ..., b_p)'$, $\mathbf{U} = (U_1, ..., U_p)'$, and $U_j + b_j$ are independently distributed $\mathcal{N}(b_j, 1)$ random variables, $j = 1, ..., p$. Thus,

$$\begin{aligned}Q &= \sum_{j=1}^{r} \lambda_j \,(U_j + b_j)^2 \;-\; \sum_{j=r+\theta+1}^{p} |\lambda_j|\, (U_j + b_j)^2 \\ &\equiv Q_1 - Q_2 \;,\end{aligned} \tag{2.3}$$

where r is the number of positive eigenvalues of $A\Sigma$ and $p - r - \theta$ is the number of negative eigenvalues of $A\Sigma$, θ being the number of null eigenvalues. Consequently, a non-central indefinite quadratic form, Q, can be expressed as a difference of independently distributed linear combinations of independent non-central chi-square random variables having one degree of freedom each. This will be referred to as a general linear combination of such variables. It should be noted that the chi-square random variables are central whenever $\boldsymbol{\mu} = \mathbf{0}$. When $A \geq 0$, Q is a positive semidefinite quadratic form, and $Q \sim Q_1$ as defined in Equation (2.3). We note that if A is not symmetric, it suffices to replace this matrix by $(A+A')/2$, which results in the same quadratic form. Accordingly, it will be assumed without any loss of generality that the matrices of the quadratic forms being considered are symmetric.

As shown in Mathai and Provost (1992), the s^{th} cumulant of $\mathbf{X}'A\mathbf{X}$ where $\mathbf{X} \sim \mathcal{N}_p(\boldsymbol{\mu}, \Sigma)$ is

$$k(s) = 2^{s-1}s!\Big(\mathrm{tr}(A\Sigma)^s/s + \boldsymbol{\mu}'(A\Sigma)^{s-1}A\boldsymbol{\mu}\Big)\,, \tag{2.4}$$

$\mathrm{tr}(\cdot)$ denoting the trace of $(\cdot)$. It should be noted that $\mathrm{tr}(A\Sigma)^s = \sum_{j=1}^{p} \lambda_j^s$ where the λ_j's, $j=1, ..., p$, are the eigenvalues of $A\Sigma$. The moments of a random variable can be obtained from its cumulants by means of a recursive relationship that is derived for instance in Smith (1995). Accordingly, the h^{th} moment of $\mathbf{X}'A\mathbf{X}$ is given by

$$\mu(h) = \sum_{i=0}^{h-1} \frac{(h-1)!}{(h-1-i)!\,i!}\, k(h-i)\,\mu(i)\ , \tag{2.5}$$

where $k(s)$ is as specified by Equation (2.4).

One can make use of Equation (2.5) to determine the moments of the positive definite quadratic forms, $Q_1 \equiv \mathbf{W}_1' A_1 \mathbf{W}_1$ and $Q_2 \equiv \mathbf{W}_2' A_2 \mathbf{W}_2$, appearing in Equation (3) where $A_1 = \mathrm{diag}(\lambda_1, ..., \lambda_r)$, $A_2 = \mathrm{diag}(|\lambda_{r+\theta+1}|, ..., |\lambda_p|)$, $\mathbf{W}_1 \sim \mathcal{N}_r(\boldsymbol{b_1}, I)$ with $\boldsymbol{b_1} = (b_1, ..., b_r)'$, and $\mathbf{W}_2 \sim \mathcal{N}_{p-r-\theta}(\boldsymbol{b_2}, I)$ with $\boldsymbol{b_2} = (b_{r+\theta+1}, ..., b_p)'$, the b_j's being as defined in Equation (2.2).

Since an indefinite quadratic form is distributed as the difference of two positive definite quadratic forms, its density function can be obtained via the transformation of variables technique. For the problem at hand, letting $h_Q(q)$, $f_{Q_1}(q_1)$ and $f_{Q_2}(q_2)$ respectively denote the approximate densities of Q, Q_1 and Q_2, the approximate density function of the indefinite quadratic form Q is given by

$$h_Q(q) = \begin{cases} h_P(q) & \text{for } q \geq 0\ , \\ h_N(q) & \text{for } q < 0\ , \end{cases} \tag{2.6}$$

where

$$h_P(q) = \int_0^\infty f_{Q_1}(q+x)\, f_{Q_2}(x)\,\mathrm{d}x\ , \tag{2.7}$$

$$h_N(q) = \int_{-q}^{\infty} f_{Q_1}(q+x)\, f_{Q_2}(x)\, \mathrm{d}x \ , \tag{2.8}$$

and $h_P(q)$ and $h_N(q)$ are explicitly given in the next section.

3. LAGUERRE POLYNOMIAL DENSITY APPROXIMANTS

In order to approximate the distribution of a positive definite quadratic form, it is appropriate to make use of an approximation that is based on Laguerre polynomials since their associated weight function is proportional to a gamma density function with parameters $\alpha \equiv \nu+1$ and $\beta = 1$. Accordingly, letting Y be a gamma-type random variable whose exact raw moments are denoted by $\mu_Y(h)$, $h = 0, 1, ..., d$, we first approximate the distribution of $X = Y/\beta$ where β, the second parameter of the gamma approximation, which can be easily obtained by matching moments, is given by

$$\beta = \frac{\mu_Y(2)}{\mu_Y(1)} - \mu_Y(1) \ . \tag{3.1}$$

Similarly, the shape parameter ν in the weight function is determined as follows:

$$\nu = \frac{\mu_Y^2(1)}{\mu_Y(2) - \mu_Y^2(1)} - 1 \ . \tag{3.2}$$

Let $L_i^\nu(x)$ denote the i^{th} degree Laguerre polynomial with parameter ν, that is,

$$L_i^\nu(x) = \sum_{k=0}^{i} d_{i,k}^\nu\, x^k \tag{3.3}$$

where

$$d_{i,k}^\nu = \frac{(-1)^{i-k}\, \Gamma(i+\nu+1)}{(i-k)!\, k!\, \Gamma(\nu+k+1)} \ . \tag{3.4}$$

As explained in Provost and Ha (2009), on equating $\int_0^\infty L_i^\nu(y)\, f(y)\, \mathrm{d}y$ to $\int_0^\infty L_i^\nu(y)\, f_{Y_d}(y)\, \mathrm{d}y$ for $h = 0, 1, ..., d$, where $f(y)$ is the exact density function being approximated and $f_{Y_d}(x)$ denotes the representation of the approximate density function given in Equation (3.19) (which, incidentally, is equivalent to assuming that the first d moments of the approximate distribution coincide with those of the target distribution), one can determine the coefficients of the Laguerre polynomials by making use of their orthogonality property. Then, by collecting the coefficients of each monomial x^k in the resulting representation, one can express the d^{th} degree Laguerre polynomial density approximant as

$$g_{X_d}(x) = c_\nu\, w_\nu(x) \sum_{k=0}^{d} \xi_{\nu,k}\, x^k \ , \tag{3.5}$$

where

$$c_\nu \;=\; 1/\Gamma(\nu+1)\;, \tag{3.6}$$

$$w_\nu(x) \;=\; x^\nu e^{-x}\;, \tag{3.7}$$

and the coefficients $\xi_{\nu,k}$ can be obtained as

$$\xi_{\nu,k} \;=\; \begin{cases} 1+\displaystyle\sum_{i=2}^{d}\eta_i^\nu d_{i,k}^\nu\,, & \text{for } k=0\;; \\ \displaystyle\sum_{i=2}^{d}\eta_i^\nu d_{i,k}^\nu\,, & \text{for } k=1\;, \\ \displaystyle\sum_{i=k}^{d}\eta_i^\nu d_{i,k}^\nu\,, & \text{for } k=2,...,d\;, \end{cases} \tag{3.8}$$

with

$$\eta_i^\nu \;=\; \frac{i!}{\Gamma(\nu+i+1)}\sum_{k=0}^{i} d_{i,k}^\nu\,\mu_X(k) \tag{3.9}$$

and

$$\mu_X(k) \;=\; \mu_Y(k)/\beta^k\;. \tag{3.10}$$

Thus, the representation of the approximate density function given in Equation (3.5) can be viewed as a mixture of $d+1$ gamma densities with parameters $\nu+k+1$ and 1. The density function of the random variable Y can then be obtained from $g_{X_d}(x)$ as specified in Equation (3.5) via the transformation $Y=\beta X$ as

$$f_{Y_d}(y) \;=\; g_{X_d}(y/\beta)/\beta\;. \tag{3.11}$$

This form of the approximate density function lends itself more readily to algebraic manipulations than that specified in Equation (3.19), which may be somewhat simpler to evaluate.

The corresponding approximate cumulative distribution function of Y evaluated at $c_0>0$ is then

$$\begin{aligned} F_{Y_d}(c_0) \;&=\; \int_0^{c_0} g_{X_d}(y/\beta)/\beta\,\mathrm{d}y \\ &=\; \int_0^{c_0/\beta} g_{X_d}(x)\,\mathrm{d}x \\ &=\; \int_0^{c_0/\beta} c_\nu\,w_\nu(x)\sum_{k=0}^{d}\xi_{\nu,k}\,x^k\,\mathrm{d}t \\ &=\; \sum_{i=0}^{d}\xi_{\nu,i}\,\frac{\Gamma(\nu+i+1)-\Gamma(\nu+i+1,c_0/\beta)}{\Gamma(\nu+1)}\;, \end{aligned} \tag{3.12}$$

where

$$\Gamma(a,\theta) = \int_\theta^\infty t^{a-1} e^{-t}\,\mathrm{d}t \tag{3.13}$$

denotes the incomplete gamma function. Conditions ensuring that the proposed approximants, whether applied to quadratic forms or random variables having an asymptotic chi-square distribution, will converge to their exact density functions, are available in Alexits (1961, p. 304).

The density functions of Q_1 and Q_2 as defined in Equation (2.3) can be approximated from their respective moments which can be determined in Equation (2.5). The density of an indefinite quadratic form $Q = Q_1 - Q_2$, where Q_1 and Q_2 are positive definite quadratic forms, can then be approximated by making use of Equation (2.6) where $f_{Q_1}(\cdot)$ and $f_{Q_2}(\cdot)$ respectively denote the Laguerre polynomial density approximants of Q_1 and Q_2, which are available from Equation (3.11). Explicit representations of $h_P(q)$ and $h_N(q)$ as specified by Equations (2.7) and (2.8), respectively, can be obtained as follows. When q is positive, the probability density function of Q is given by

$$\begin{aligned} h_P(q) &= \int_0^\infty f_{Q_1}(q+y)\, f_{Q_2}(y)\,\mathrm{d}y \\ &= \int_0^\infty \Big(\gamma_{\nu_1,\beta_1}(q+y) \sum_{i=0}^{d} \xi_{\nu_1,i} \Big(\frac{q+y}{\beta_1}\Big)^i\Big) \Big(\gamma_{\nu_2,\beta_2}(y) \sum_{j=0}^{d} \xi_{\nu_2,j} \Big(\frac{y}{\beta_2}\Big)^j\Big)\,\mathrm{d}y \end{aligned} \tag{3.14}$$

with $\gamma_{\nu_\ell,\beta_\ell}(z) = z^{\nu_\ell}\, e^{-z/\beta_\ell}/(\beta_\ell^{\nu_\ell+1}\,\Gamma(\nu_\ell+1))$, $\ell = 1,2$; ν_ℓ and β_ℓ determined from Equations (3.1) and (3.2), respectively, $\ell = 1,2$, the coefficients $\xi_{\nu_1,i}$ and $\xi_{\nu_2,i}$ being as defined in Equation (3.8). Identities 3.384 3 and 9.220 4 from Gradshteyn and Ryzhik (1980) yield

$$\begin{aligned} h_P(q) &= \sum_{i=0}^{d}\sum_{j=0}^{d} \xi_{\nu_1,i}\,\xi_{\nu_2,j} \int_0^\infty \Big(\frac{q+y}{\beta_1}\Big)^i \Big(\frac{y}{\beta_2}\Big)^j \gamma_{\nu_1,\beta_1}(q+y)\,\gamma_{\nu_2,\beta_2}(y)\,\mathrm{d}y \\ &= \sum_{i=0}^{d}\sum_{j=0}^{d} \frac{\xi_{\nu_1,i}\,\xi_{\nu_2,j}\, e^{-q/\beta_1}}{\beta_1^{\nu_1+i+1}\,\beta_2^{\nu_2+j+1}\,\Gamma(\nu_1+1)\,\Gamma(\nu_2+1)} \Bigg(\Big(\frac{\beta_1+\beta_2}{\beta_1\beta_2}\Big)^{-1-i-j-\nu_1-\nu_2} \\ &\quad \times \Gamma\big(i+j+\nu_1+\nu_2+1\big)\, {}_1F_1\big(-i-\nu_1, -i-j-\nu_1-\nu_2, q(\beta_1+\beta_2)/(\beta_1\beta_2)\big) \\ &\quad + \frac{\Gamma\big(-1-i-j-\nu_1-\nu_2\big)\,\Gamma(j+\nu_2+1)}{\Gamma(-i-\nu_1)}\, q^{i+j+\nu_1+\nu_2+1} \\ &\quad \times {}_1F_1\big(j+\nu_2+1,\, i+j+\nu_1+\nu_2+2,\, q(\beta_1+\beta_2)/(\beta_1\beta_2)\big)\Bigg), \end{aligned} \tag{3.15}$$

where

$$ {}_1F_1(a,b,z) = \sum_{k=0}^{\infty} \frac{\Gamma(a+k)\,\Gamma(b)\,z^k}{\Gamma(a)\,\Gamma(b+k)\,k!}$$

is Kummer's confluent hypergeometric function. Similarly, when q is negative, one has

$$\begin{aligned}
h_N(q) &= \int_{-q}^{\infty}\Big(\gamma_{\nu_1,\beta_1}(q+y)\sum_{i=0}^{d}\xi_{\nu_1,i}\Big(\frac{q+y}{\beta_1}\Big)^i\Big)\Big(\gamma_{\nu_2,\beta_2}(y)\sum_{j=0}^{d}\xi_{\nu_2,j}\Big(\frac{y}{\beta_2}\Big)^j\Big)\,\mathrm{d}y\\
&= \int_{0}^{\infty}\Big(\gamma_{\nu_1,\beta_1}(w)\sum_{i=0}^{d}\xi_{\nu_1,i}\Big(\frac{w}{\beta_1}\Big)^i\Big)\Big(\gamma_{\nu_2,\beta_2}(w-q)\sum_{j=0}^{d}\xi_{\nu_2,j}\Big(\frac{w-q}{\beta_2}\Big)^j\Big)\,\mathrm{d}w\\
&= \sum_{i=0}^{d}\sum_{j=0}^{d}\frac{\xi_{\nu_1,i}\,\xi_{\nu_2,j}}{\beta_1^i\,\beta_2^j}\int_0^{\infty} w^i\,(w-q)^j\,\gamma_{\nu_1,\beta_1}(w)\,\gamma_{\nu_2,\beta_2}(w-q)\,\mathrm{d}w\\
&= \sum_{i=0}^{d}\sum_{j=0}^{d}\frac{\xi_{\nu_1,i}\,\xi_{\nu_2,j}\,e^{q/\beta_2}}{\beta_1^{\nu_1+i+1}\,\beta_2^{\nu_2+j+1}\,\Gamma(\nu_1+1)\,\Gamma(\nu_2+1)}\Big(\Big(\frac{\beta_1+\beta_2}{\beta_1\,\beta_2}\Big)^{-1-i-j-\nu_1-\nu_2}\\
&\quad\times\Gamma\big(i+j+\nu_1+\nu_2+1\big)\,{}_1F_1\big(-j-\nu_2,-i-j-\nu_1-\nu_2,-q(\beta_1+\beta_2)/(\beta_1\,\beta_2)\big)\\
&\quad+\frac{\Gamma(-1-i-j-\nu_1-\nu_2)\,\Gamma(i+\nu_1+1)}{\Gamma(-j-\nu_2)}\,(-q)^{i+j+\nu_1+\nu_2+1}\\
&\quad\times\,{}_1F_1\big(i+\nu_1+1,\,i+j+\nu_1+\nu_2+2,\,-q(\beta_1+\beta_2)/(\beta_1\,\beta_2)\big)\Big)\,. \qquad (3.16)
\end{aligned}$$

We note that the series representations given in Equations (3.15) and (3.16) do not converge when ν_1 or ν_2 are integer-valued. In this case, one would have to evaluate the integral representations by numerical integration.

The corresponding cumulative distribution function is then obtained by integration. The approximate cumulative distribution function for the negative part of Q is given by

$$\begin{aligned}
H_N(y) &= \int_{-\infty}^{y} h_N(q)\,\mathrm{d}q\\
&= \sum_{i=0}^{d}\sum_{j=0}^{d}\frac{\xi_{\nu_1,i}\,\xi_{\nu_2,j}}{\beta_1^{\nu_1+i+1}\,\beta_2^{\nu_2+j+1}\,\Gamma(\nu_1+1)\,\Gamma(\nu_2+1)}\Big(\Big(\frac{\beta_1+\beta_2}{\beta_1\,\beta_2}\Big)^{-1-i-j-\nu_1-\nu_2}\\
&\quad\times\Gamma(i+j+\nu_1+\nu_2+1)\\
&\quad\times\int_{-\infty}^{y} e^{q/\beta_2}\,{}_1F_1\big(-j-\nu_2,-i-j-\nu_1-\nu_2,-q(\beta_1+\beta_2)/(\beta_1\,\beta_2)\big)\,\mathrm{d}q \qquad (3.17)\\
&\quad+\frac{\Gamma(-1-i-j-\nu_1-\nu_2)\,\Gamma(i+\nu_1+1)}{\Gamma(-j-\nu_2)}\\
&\quad\times\int_{-\infty}^{y}(-q)^{i+j+\nu_1+\nu_2+1}\,e^{q/\beta_2}\\
&\qquad\times\,{}_1F_1\big(i+\nu_1+1,\,i+j+\nu_1+\nu_2+2,\,-q(\beta_1+\beta_2)/(\beta_1\,\beta_2)\big)\,\mathrm{d}q\Big) =
\end{aligned}$$

$$= \sum_{i=0}^{d}\sum_{j=0}^{d}\sum_{k=0}^{\infty} \frac{\xi_{\nu_1,i}\,\xi_{\nu_2,j}\,(\beta_1+\beta_2)^k}{\beta_1^{\nu_1+k+i+1}\,\beta_2^{\nu_2+k+j+1}\,\Gamma(\nu_1+1)\,\Gamma(\nu_2+1)} \Bigg(\Big(\frac{\beta_1+\beta_2}{\beta_1\,\beta_2}\Big)^{-1-i-j-\nu_1-\nu_2}$$
$$\times\,\Gamma(1+i+j+\nu_1+\nu_2)\,\frac{\Gamma(-j+k-\nu_2)\,\Gamma(-i-j-\nu_1-\nu_2)}{\Gamma(-j-\nu_2)\,\Gamma(-i-j+k-\nu_1-\nu_2)\,k!}$$
$$\times \int_{-\infty}^{y}(-q)^k\,e^{q/\beta_2}\mathrm{d}q + \frac{\Gamma(-1-i-j-\nu_1-\nu_2)}{\Gamma(-j-\nu_2)}$$
$$\times\,\frac{\Gamma(i+k+\nu_1+1)\,\Gamma(i+j+\nu_1+\nu_2+2)}{\Gamma(i+j+k+\nu_1+\nu_2+2)\,k!}\int_{-\infty}^{y}(-q)^{i+j+k+\nu_1+\nu_2+1}\,e^{q/\beta_2}\,\mathrm{d}q\Bigg)$$

(3.17)
$$= \sum_{i=0}^{d}\sum_{j=0}^{d}\sum_{k=0}^{\infty} \frac{\xi_{\nu_1,i}\,\xi_{\nu_2,j}\,(\beta_1+\beta_2)^k}{\beta_1^{\nu_1+k+i+1}\,\beta_2^{\nu_2+k+j+1}\,\Gamma(\nu_1+1)\,\Gamma(\nu_2+1)} \Bigg(\Big(\frac{\beta_1+\beta_2}{\beta_1\,\beta_2}\Big)^{-1-i-j-\nu_1-\nu_2}$$
$$\times\,\Gamma(1+i+j+\nu_1+\nu_2)\frac{\Gamma(-j+k-\nu_2)\,\Gamma(-i-j-\nu_1-\nu_2)\,\beta_2^{k+1}\,\Gamma(k+1,-y/\beta_2)}{\Gamma(-j-\nu_2)\,\Gamma(-i-j+k-\nu_1-\nu_2)\,k!}$$
$$+\,\frac{\Gamma(-1-i-j-\nu_1-\nu_2)}{\Gamma(-j-\nu_2)}\,\frac{\Gamma(i+k+\nu_1+1)\,\Gamma(i+j+\nu_1+\nu_2+2)}{\Gamma(i+j+k+\nu_1+\nu_2+2)\,k!}$$
$$\times\,\beta_2^{i+j+k+\nu_1+\nu_2+2}\,\Gamma(i+j+k+\nu_1+\nu_2+2,-y/\beta_2)\Bigg)\,.$$

Similarly, the approximate cumulative distribution function for the positive part of Q can be expressed as follows:

$$H_P(y) \;=\; H_N(0) + \int_0^y h_P(q)\,\mathrm{d}q$$
$$=\; H_N(0) + \sum_{i=0}^{d}\sum_{j=0}^{d} \frac{\xi_{\nu_1,i}\,\xi_{\nu_2,j}}{\beta_1^{\nu_1+i+1}\,\beta_2^{\nu_2+j+1}\,\Gamma(\nu_1+1)\,\Gamma(\nu_2+2)}$$
$$\times\,\Bigg(\Big(\frac{\beta_1+\beta_2}{\beta_1\,\beta_2}\Big)^{-1-i-j-\nu_1-\nu_2}\,\Gamma(i+j+\nu_1+\nu_2+1)$$
$$\times \int_0^y e^{-q/\beta_1}\,{}_1F_1\big(-i-\nu_1,\,-i-j-\nu_1-\nu_2,\,q(\beta_1+\beta_2)/(\beta_1\,\beta_2)\big)\,\mathrm{d}q$$
$$+\,\frac{\Gamma(-1-i-j-\nu_1-\nu_2)\,\Gamma(j+\nu_2+1)}{\Gamma(-i-\nu_1)}$$
$$\times \int_0^y q^{i+j+\nu_1+\nu_2+1}\,e^{-q/\beta_1}\,{}_1F_1\big(j+\nu_2+1, i+j+\nu_1+\nu_2+2, q(\beta_1+\beta_2)/(\beta_1\,\beta_2)\big)\,\mathrm{d}q\Bigg)$$

(3.18)
$$=\; H_N(0) + \sum_{i=0}^{d}\sum_{j=0}^{d}\sum_{k=0}^{\infty} \frac{\xi_{\nu_1,i}\,\xi_{\nu_2,j}}{\beta_1^{\nu_1+i+1}\,\beta_2^{\nu_2+j+1}\,\Gamma(\nu_1+1)\,\Gamma(\nu_2+1)}$$
$$\times\Bigg(\Big(\frac{\beta_1+\beta_2}{\beta_1\,\beta_2}\Big)^{-1-i-j+k-\nu_1-\nu_2}\,\frac{\Gamma(1+i+j+\nu_1+\nu_2)\,\Gamma(-i-\nu_1+k)\,\Gamma(-i-j-\nu_1-\nu_2)}{\Gamma(-i-\nu_1)\,\Gamma(-i-j+k-\nu_1-\nu_2)\,k!}$$
$$\times\,\beta_1^{k+1}\Big(\Gamma(1+k)-\Gamma(1+k,y/\beta_1)\Big)$$
$$+\,\frac{\Gamma(-1-i-j-\nu_1-\nu_2)}{\Gamma(-i-\nu_1)}\,\frac{\Gamma(j+k+\nu_2+1)\,\Gamma(i+j+\nu_1+\nu_2+2)}{\Gamma(i+j+k+\nu_1+\nu_2+2)\,k!}$$
$$\times\,\beta_1^{i+j+k+\nu_1+\nu_2+2}\Big(\Gamma(i+j+k+\nu_1+\nu_2+2)-\Gamma(i+j+k+\nu_1+\nu_2+2,y/\beta_1)\Big)\Bigg)$$

where

$$H_N(0) = \sum_{i=0}^{d}\sum_{j=0}^{d}\sum_{k=0}^{\infty} \frac{\xi_{\nu_1,i}\,\xi_{\nu_2,j}\,(\beta_1+\beta_2)^k}{\beta_1^{\nu_1+k+i+1}\,\beta_2^{\nu_2+k+j+1}\,\Gamma(\nu_1+1)\,\Gamma(\nu_2+1)}$$
$$\times \Bigg(\Big(\frac{\beta_1+\beta_2}{\beta_1\,\beta_2}\Big)^{-1-i-j-\nu_1-\nu_2}\Gamma(1+i+j+\nu_1+\nu_2)$$
$$\times \frac{\Gamma(-j+k-\nu_2)\,\Gamma(-i-j-\nu_1-\nu_2)\,\beta_2^{k+1}\,\Gamma(k+1)}{\Gamma(-j-\nu_2)\,\Gamma(-i-j+k-\nu_1-\nu_2)\,k!}$$
$$+\ \frac{\Gamma(-1-i-j-\nu_1-\nu_2)}{\Gamma(-j-\nu_2)}\ \frac{\Gamma(i+k+\nu_1+1)\,\Gamma(i+j+\nu_1+\nu_2+2)}{k!}$$
$$\times\ \beta_2^{i+j+k+\nu_1+\nu_2+2}\Bigg)\,.$$

Even though the sum over k has infinitely many summands, we observed that fifty terms provide sufficient accuracy. In most cases of interest, a suitable degree for a density approximation can be determined by a *de visu* inspection of the density plots of approximants of successive degrees. More specifically, one might be satisfied that an approximant of degree $d+1$ is adequate if no noticeable differences are observed when comparing the plots of approximants of degrees d and $d+2$. This criterion was applied to all the examples presented in Section 5. Equivalently, one may wish to set a tolerance for the integrated absolute difference of approximants of successive degrees and select the number of terms to be used in the approximation accordingly. If one wishes to determine the number of terms required to obtain a satisfactory approximation for a specific percentile, one could evaluate the percentile approximations for successive values of d until convergence is observed or a preset tolerance value exceeds the difference of two successive approximations. Since we are dealing with a sequence of approximants converging to the exact density function, the close proximity of successive approximants indicates that convergence is nearly attained. Bounds for the integrated absolute and squared truncation errors are obtained in the remainder of this section. In light of Equation (3.19) of Provost (2005), the truncated density function corresponding to that given in Equation (3.11) can be expressed as

$$f_{Y_d}(y) = \frac{y^{\nu}e^{-y/\beta}}{\beta^{\nu+1}}\sum_{j=0}^{d}\delta_j^{\nu}L_j^{\nu}(y/\beta) \tag{3.19}$$

with $L_j^{\nu}(\cdot)$ as defined in Equation (11) and

$$\delta_j^{\nu} = \sum_{k=0}^{j}\frac{(-1)^k\,j!\,\mu_X(j-k)}{k!\,(j-k)!\,\Gamma(\nu+j-k+1)}\,.$$

Let $F_{Y_d}(y)$ and $F_Y(y)$ respectively denote the cumulative distribution functions of Y_d and Y and $f_Y(y)$ denote the density function being approximated.

Letting

$$\delta_j^\nu = \frac{j!}{\Gamma(\nu+j+1)}\;\psi_j^\nu$$

where

$$\psi_j^\nu = \sum_{k=0}^{j} \frac{(-1)^k\,\Gamma(\nu+j+1)\,\mu_X(j-k)}{k!\,(j-k)!\,\Gamma(\nu+j-k+1)}\;,$$

a bound for the truncation error with respect to the probability density function of Y can be determined as follows:

$$\begin{aligned}\mathcal{E}_d(y) &= \big|f_Y(y) - f_{Y_d}(y)\big| \\ &= \frac{y^\nu e^{-y/c}}{c^{\nu+1}} \sum_{j=d+1}^{\infty} \frac{j!}{\Gamma(\nu+j+1)}\;|\psi_j^\nu|\,|L_j(\nu, y/c)|\;,\end{aligned} \tag{3.20}$$

where according to Szegö (1975),

$$\begin{aligned}L_j(\nu, y/c) &\le \frac{(\nu+1)_j}{j!}\;e^{y/(2c)} \\ &= \frac{\Gamma(\nu+1+j)}{\Gamma(\nu+1)\,j!}\;e^{y/(2c)}\;.\end{aligned} \tag{3.21}$$

Thus,

$$\mathcal{E}_d(y) \le \frac{y^\nu e^{-y/(2c)}}{c^{\nu+1}\,\Gamma(\nu+1)} \sum_{j=d+1}^{\infty} |\psi_j^\nu|\;, \tag{3.22}$$

and letting $\lambda_d = \sum_{j=d+1}^{\infty} |\psi_j^\nu|$, a bound for e_d, the integrated absolute truncation error, can be obtained as follows:

$$\begin{aligned}e_d &= \int_0^\infty \mathcal{E}_d(y)\,\mathrm{d}y \\ &\le \int_0^\infty \lambda_d\,\frac{y^\nu e^{-y/(2c)}}{c^{\nu+1}\,\Gamma(\nu+1)}\;\mathrm{d}y \\ &= 2^{\nu+1}\lambda_d \\ &= 2^{\nu+1} \sum_{j=d+1}^{\infty} \left| \sum_{k=0}^{j} \frac{(-1)^k\,\Gamma(\nu+j+1)\,\mu_X(j-k)}{k!\,(j-k)!\,\Gamma(\nu+j-k+1)} \right|.\end{aligned} \tag{3.23}$$

This result yields a bound for the distribution function integrated absolute error:

$$\begin{aligned}|F_Y(y) - F_{Y_d}(y)| &= \left| \int_0^y \big(f_Y(y) - f_{Y_d}(y)\big)\,\mathrm{d}x \right| \\ &\le \int_0^\infty \big|f_Y(y) - f_{Y_d}(y)\big|\,\mathrm{d}x \\ &\le 2^{\nu+1}\lambda_d\;.\end{aligned} \tag{3.24}$$

A bound for the density function integrated squared error can be similarly obtained:

$$
\begin{aligned}
e_d^* &= \int_0^\infty \mathcal{E}_d^2(y)\,\mathrm{d}y \\
&\le \int_0^\infty \lambda_d^2 \,\frac{y^{2\nu} e^{-y/c}}{c^{2(\nu+1)}\,\Gamma^2(\nu+1)}\,\mathrm{d}y \qquad (3.25)\\
&= \frac{\lambda_d^2\,\Gamma(2\nu+1)}{c\,\Gamma^2(\nu+1)}\;.
\end{aligned}
$$

Admittedly, these bounds are not very tight. Moreover, a precise order of convergence cannot be determined since these error bounds depend on the moments of the distribution being approximated.

4. THE ALGORITHM

The following algorithm can be utilized to approximate the density function of the quadratic form $Q = \mathbf{X}'A\mathbf{X}$ where $\mathbf{X} \sim \mathcal{N}_p(\boldsymbol{\mu}, \Sigma)$, $\Sigma > 0$ and A is an indefinite symmetric real matrix.

1. The eigenvalues of $A\Sigma$ denoted by $\lambda_1 \ge \cdots \ge \lambda_r > 0 > \lambda_{r+\theta+1} \ge \cdots \ge \lambda_p$, and the *corresponding normalized* eigenvectors, $\boldsymbol{\nu}_1, ..., \boldsymbol{\nu}_p$, are determined.

2. Letting $P = (\boldsymbol{\nu}_1, ..., \boldsymbol{\nu}_p)$, $\gamma_1, ..., \gamma_p$ be the eigenvalues of Σ, $\mathbf{t}_1, ..., \mathbf{t}_p$ be the *normalized* eigenvectors of Σ corresponding to $\gamma_1, ..., \gamma_p$, $T = (\mathbf{t}_1, ..., \mathbf{t}_p)$, $\Sigma^{-1/2} = T\,\mathrm{diag}(\gamma_1^{-1/2}, ..., \gamma_p^{-1/2})\;\; T'$, $\boldsymbol{b} = (b_1, \cdots, b_p)' = P'\,\Sigma^{-1/2}\,\boldsymbol{\mu}$ and the U_j's denote independently distributed standard normal variables, one has the decomposition $Q = \sum_{j=1}^r \lambda_j (U_j + b_j)^2 - \sum_{j=r+\theta+1}^p |\lambda_j| (U_j + b_j)^2 \equiv Q_1 - Q_2$, where $Q_1 \equiv \mathbf{W}_1' A_1 \mathbf{W}_1$, $\mathbf{W}_1 \sim \mathcal{N}_r(\boldsymbol{b_1}, I)$, $\boldsymbol{b_1} = (b_1, ..., b_r)'$, $A_1 = \mathrm{diag}(\lambda_1, ..., \lambda_r)$, and $Q_2 \equiv \mathbf{W}_2' A_2 \mathbf{W}_2$, $\mathbf{W}_2 \sim \mathcal{N}_{p-r-\theta}(\boldsymbol{b_2}, I)$, $\boldsymbol{b_2} = (b_{r+\theta+1}, ..., b_p)'$, $A_2 = \mathrm{diag}(|\lambda_{r+\theta+1}|, ..., |\lambda_p|)$. Clearly, $\mathbf{b} = \mathbf{0}$ whenever $\boldsymbol{\mu} = \mathbf{0}$ and, in that case, there is no need to determine the matrices P or T.

3. The cumulants and the moments of Q_1 and Q_2 are obtained from Equations (2.4) and (2.5), respectively.

4. Laguerre polynomial density approximants, as specified by Equation (3.11), are obtained for each of the positive definite quadratic form Q_1 and Q_2 on the basis of their respective moments and denoted by $f_{Q_1}(\cdot)$ and $f_{Q_2}(\cdot)$. This requires the determination of β_i and ν_i from Equations (3.1) and (3.2) for each Q_i, $i = 1, 2$. The degree d of a given approximant can initially be set equal to 6 and then progressively increased until convergence is observed (graphically or with respect to certain percentiles of interest).

5. Given $f_{Q_1}(\cdot)$ and $f_{Q_2}(\cdot)$, the approximate density of Q is obtained from Equation (2.6) where $h_P(\cdot)$ and $h_N(\cdot)$ are respectively specified by Equation (3.15) and (3.16).

6. The corresponding cumulative distribution function can then be evaluated from Equations (3.17) and (3.18).

Remarks. In the case of a nonnegative definite quadratic form, that is, $Q = \mathbf{X}'A\mathbf{X}$ where $A = A'$ and $A \geq 0$, all the eigenvalues of A are nonnegative and one has $Q = Q_1$ whose approximate density and distribution functions are directly obtained from Equations (3.11) and (3.12), respectively.

5. NUMERICAL EXAMPLES

In this section, the proposed Laguerre polynomial approximation methodology is applied to positive definite and indefinite quadratic forms as well as the Durbin–Watson statistic. In each case, the approximated distribution will either be compared with the exact or simulated distributions. It should be noted that Equations (3.15) and (3.16) can be viewed as closed form representations since the ${}_1F_1$ hypergeometric function can be readily evaluated by most mathematical or statistical computing packages. More precision can be obtained by increasing the degree d of the polynomial adjustment appearing in Equation (13). However, when several successive approximations are seen to be nearly identical, the gain in accuracy becomes minimal. Percentage points were obtained by equating the distribution functions to a given probability and solving the resulting equations numerically. The simulated distribution functions were generated by making use of the Monte Carlo technique.

Example 1.

We first consider the case of a positive definite central quadratic form in independently distributed standard normal variables, which, according to Equation (2.2), can be expressed as

$$Q^I = \mathbf{X}'A\mathbf{X} = \sum_{j=1}^{r} \lambda_j Y_j \ , \tag{5.1}$$

where $A > 0$, $\mathbf{X} \sim \mathcal{N}_p(\mathbf{0}, I)$, λ_j, $j = 1, ..., r$, are the positive eigenvalues of A, the Y_j's, $j = 1, ..., r$ are independently distributed central chi-square random variables, each having one degree of freedom.

In this first example, $\lambda_1 = \lambda_2 = 1$, $\lambda_3 = \lambda_4 = 2.5$, and $\lambda_5 = \lambda_6 = 9$. Since the eigenvalues occur in pairs, the exact density function can be determined from the positive part of Equation (3.23) wherein $\lambda'_k = \lambda_{k/2}$, $s = t = r/2$, $\rho = 0$ and an empty product is interpreted as 1. In this case, with $\nu = 0.77054$ and $\beta = 14.12$, the density function of Q^I can be directly approximated by means of Equation (3.11) in conjunction with Equations (3.1), (3.2) and (3.5). Certain quantiles determined from the exact distribution, the gamma density and the sixth and fourteenth-degree Laguerre polynomial approximant specified by Equation (3.12) are included in Table 1.

The 95^{th} percentiles obtained from approximants of degrees 4, 6, 8, 10, 12 and 14 are respectively 60.5291, 62.5418, 62.3713, 61.8045, 61.7053 and 61.8384. This sequence suggests that a fourteenth-degree approximant might be sufficiently accurate. The exact 95^{th} percentile is in fact 61.8999. Certain extreme tail quantiles obtained from the exact density function and the fourteenth-degree Laguerre polynomial approximants are presented in Table 2. Bounds for the integrated absolute and squared errors are plotted in Figure 1 for various values of d. Figure 2 shows exact integrated absolute (left panel) and squared (right panel) differences between the exact and approximate cumulative distribution function versus d.

Table 1: Certain quantiles of Q^I.

CDF	Gamma	Laguerre ($d = 6$)	Laguerre ($d = 14$)	Exact
0.01	1.43483	1.92384	2.51869	2.5795
0.05	3.77669	4.63033	5.04397	5.04193
0.10	5.88517	6.83939	7.03708	7.00919
0.50	20.4832	20.3014	20.0027	20.0400
0.90	50.0482	49.0916	49.3561	49.4183
0.95	61.6596	62.5418	61.8384	61.8999
0.99	87.6053	91.4214	90.9503	90.8707

Table 2: Certain extreme tail quantiles of Q^I.

CDF	Gamma	Laguerre ($d = 6$)	Laguerre ($d = 14$)	Exact
0.0001	0.102918	0.149769	0.403458	0.491026
0.001	0.380511	0.542588	1.00356	1.09778
0.999	123.408	127.632	132.491	132.317
0.9999	158.391	183.558	173.364	173.764

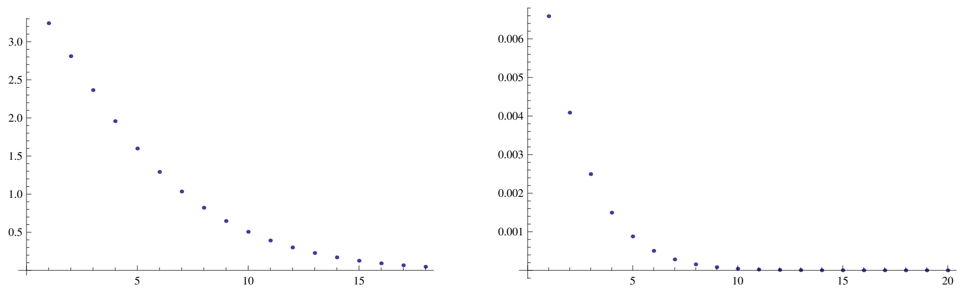

Figure 1: Bounds for the Integrated Absolute (left panel) and Squared (right panel) Truncation Errors with Respect to the Truncation Order.

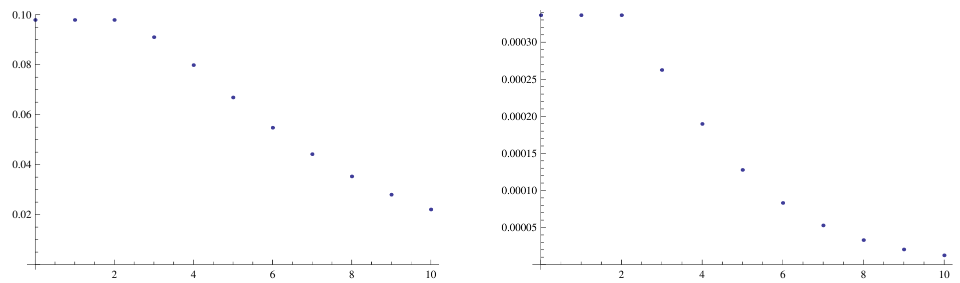

Figure 2: Integrated Absolute Difference (left panel); Integrated Squared Difference (right panel).

Example 2.

We now consider the general case of a non-central indefinite quadratic form, $Q^{II} = \mathbf{X}'A\mathbf{X}$ where

$$A = \begin{pmatrix} 1 & 1 & 2 & 6 \\ 1 & 8 & 0 & 0 \\ 2 & 0 & -1/2 & 1 \\ 6 & 0 & 1 & -2 \end{pmatrix},$$

$\mathbf{X} \sim \mathcal{N}_4(\boldsymbol{\mu}, \Sigma)$ with $\boldsymbol{\mu} = (1,2,3,4)'$ and

$$\Sigma = \begin{pmatrix} 1 & 4/5 & -1/5 & 0 \\ 4/5 & 1 & 1/3 & 1/4 \\ -1/5 & 1/3 & 1 & 0 \\ 0 & 1/4 & 0 & 1 \end{pmatrix}.$$

In light of Equation (2.3), Q^{II} can be re-expressed as

$$Q^{II} = Q_1 - Q_2 = \sum_{i=1}^{2} \lambda_i (U_i + b_i)^2 - \sum_{j=3}^{4} |\lambda_j| \, (U_j + b_j)^2 \tag{5.2}$$

where the U_i's, $i = 1,2,3,4$, are standard normal random variables, $\lambda_1 = 14.487$,

$\lambda_2 = 0.175399$, $\lambda_3 = -1.05353$, $\lambda_4 = -6.30884$, $b_1 = 3.04567$, $b_2 = 7.26373$, $b_3 = -2.10575$, and $b_4 = -2.93822$. Clearly Q^{II} can also be regarded as a general linear combination of non-central chi-square random variables. In this case, the matrices P and $\Sigma^{1/2}$ are respectively

$$P = \begin{pmatrix} 0.552559 & 0.72748 & 0.200698 & 0.353796 \\ 0.537095 & 0.0528413 & -0.257096 & -0.801647 \\ 0.119867 & -0.0192312 & -0.919478 & 0.373928 \\ 0.62597 & -0.683821 & 0.219504 & 0.303922 \end{pmatrix}$$

and

$$\Sigma^{1/2} = \begin{pmatrix} 0.829443 & 0.524395 & -0.186334 & -0.048092 \\ 0.524395 & 0.798945 & 0.248679 & 0.157654 \\ -0.186334 & 0.248679 & 0.950168 & -0.0248771 \\ -0.048092 & 0.157654 & -0.0248771 & 0.986009 \end{pmatrix}.$$

The approximate density functions of Q_1 and Q_2 were obtained by making use of sixth-degree Laguerre polynomial approximants. The resulting approximations of the density and distribution functions of Q^{II} as evaluated from Equations (3.15) and (3.16) and Equations (3.17) and (3.18) with $\nu_1 = 2.05092$, $\beta_1 = 51.8858$, $\nu_2 = 1.99611$ and $\beta_2 = 22.1952$, are plotted in Figure 3. The right panel of Figure 3 also shows the simulated distribution function, which was obtained on the basis of 100,000 replications. Accordingly, the standard error is at most $1/633 \approx 0.0016$.

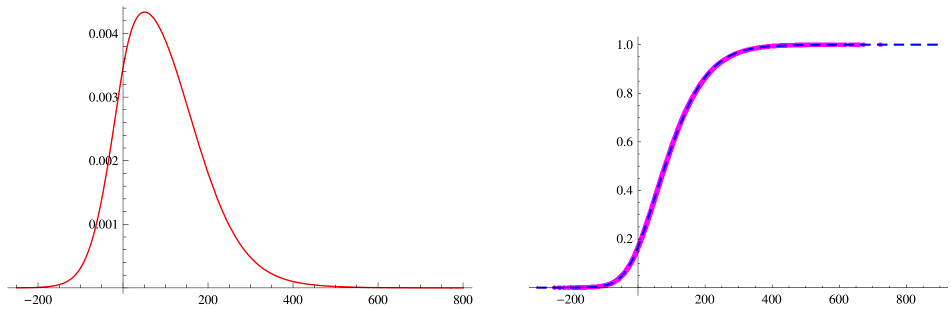

Figure 3: Approximated PDF (left panel);
Simulated and Approximated CDF (right panel).

Example 3.

Consider the following general linear combination of independently distributed central chi-square random variables:

$$Q^{III} = Q_1 - Q_2 = \sum_{i=1}^{r} \lambda_i Y_i - \sum_{j=r+\theta+1}^{p} |\lambda_j| \, Y_j \ , \tag{5.3}$$

where $\theta = 0$, the Y_j's, $j = 1, ..., 16$ are independently distributed central chi-square random variables having one degree of freedom and $\lambda_1 = \lambda_2 = 2$, $\lambda_3 = \lambda_4 = 4$, $\lambda_5 = \lambda_6 = 6$, $\lambda_7 = \lambda_8 = 8$, $\lambda_9 = \lambda_{10} = 10$, $\lambda_{11} = \lambda_{12} = -20$, $\lambda_{13} = \lambda_{14} = -30$ and $\lambda_{15} = \lambda_{16} = -40$.

Since the eigenvalues occur in pairs in the right-hand side of Equation (3.21), Q^{III} can be expressed as

$$Q^{III} = \sum_{i=1}^{s} \lambda'_i T_i - \sum_{j=s+1}^{t} |\lambda'_j| T_j \ , \tag{5.4}$$

where $s = r/2$, $t = p/2$, $\lambda'_k = \lambda_{k/2}$, $k = 1, ..., t$, and the T_i's and T_j's are independently distributed chi-square random variables, each one having *two* degrees of freedom. Imhof (1961) derived the following representation of the exact density function of Q^{III}

$$g(q) = \begin{cases} \sum_{j=1}^{s} \dfrac{\lambda_j'^{t-2} e^{-2q/(2\lambda'_j)}}{2\Big(\prod_{k=1,k\neq j}^{s}(\lambda'_j-\lambda'_k)\Big)\Big(\prod_{k=s+1}^{t}(|\lambda'_j|+|\lambda'_k|)\Big)} \ , & q \geq 0 \ , \\ \sum_{j=s+1}^{t} \dfrac{|\lambda'_j|^{t-2} e^{2q/(2|\lambda'_j|)}}{2\Big(\prod_{k=s+1,k\neq j}^{t}(|\lambda'_j|-|\lambda'_k|)\Big)\Big(\prod_{k=1}^{s}(\lambda'_j+\lambda'_k)\Big)} \ , & q < 0 \ . \end{cases} \tag{5.5}$$

The sixth-degree Laguerre polynomial density approximant of Q^{III} as determined from Equations (3.14) and (3.15) is shown in Figure 4, superimposed on the exact density.

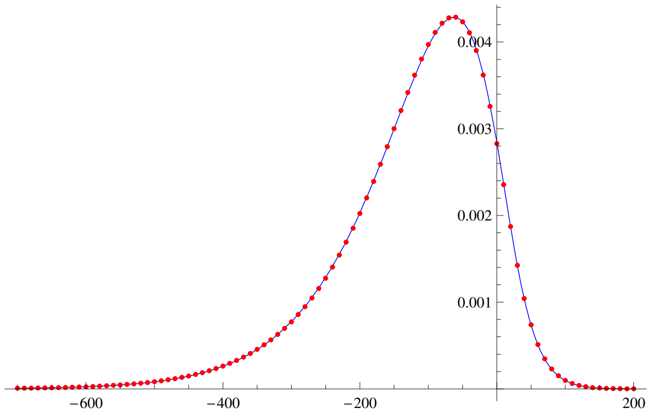

Figure 4: Exact density and Laguerre Polynomial Approximant (dotted line).

Example 4.

The statistic proposed by Durbin and Watson (1950), which in fact assesses whether the errors in the linear regression model

$$\mathbf{Y} = X\boldsymbol{\beta} + \boldsymbol{\epsilon} \tag{5.6}$$

are uncorrelated, can be expressed as

$$D = \frac{\hat{\boldsymbol{\epsilon}}' A^* \hat{\boldsymbol{\epsilon}}}{\hat{\boldsymbol{\epsilon}}'\hat{\boldsymbol{\epsilon}}} \tag{5.7}$$

where

$$\hat{\boldsymbol{\epsilon}} = \mathbf{Y} - X\hat{\boldsymbol{\beta}} \tag{5.8}$$

is the vector of residuals,

$$\hat{\boldsymbol{\beta}} = (X'X)^{-1}X'\mathbf{Y} \tag{5.9}$$

being the ordinary least-squares estimator of $\boldsymbol{\beta}$, and $A^* = (a^*_{ij})$ is a symmetric tridiagonal matrix with $a^*_{11} = a^*_{pp} = 1$; $a^*_{ii} = 2$, for $i = 2, ..., p-1$; $a^*_{ij} = -1$ if $|i-j| = 1$; and $a^*_{ij} = 0$ if $|i-j| \geq 2$. Assuming that the error vector is normally distributed, one has $\boldsymbol{\epsilon} \sim \mathcal{N}_p(\mathbf{0}, I)$ under the null hypothesis.

Then, on writing $\hat{\boldsymbol{\epsilon}}$ as $M\mathbf{Y}$ where $M_{p\times p} = I - X(X'X)^{-1}X' = M'$ is an idempotent matrix of rank $p-k$, the test statistic can be expressed as the following ratio of quadratic forms:

$$D = \frac{\mathbf{Z}'MA^*M\mathbf{Z}}{\mathbf{Z}'M\mathbf{Z}}\,, \tag{5.10}$$

where $\mathbf{Z} \sim \mathcal{N}_p(\mathbf{0}, I)$; this can be seen from the fact that $M\mathbf{Y}$ and $M\mathbf{Z}$ are identically distributed singular normal vectors with mean vector $\mathbf{0}$ and covariance matrix MM'. We note that the distribution function of D (and, in general, ratios of quadratic forms of the form $(\mathbf{X}'B\mathbf{X})/(\mathbf{X}'C\mathbf{X})$) at the point t_0 can be determined as follows:

$$\begin{aligned} \Pr(D < t_0) &= \Pr\Big(\mathbf{Z}'MA^*M\mathbf{Z} < t_0\mathbf{Z}'M\mathbf{Z}\Big) \\ &= \Pr\Big(\mathbf{Z}'M(A^*M - t_0 I)\mathbf{Z} < 0\Big)\,. \end{aligned} \tag{5.11}$$

On letting $U = \mathbf{Z}'M(A^*M - t_0I)\mathbf{Z}$, U can be re-expressed as a difference of two positive quadratic forms by applying Steps 1 and 2 of the algorithm provided in Section 4, with $A = M(A^*M - t_0I)$, $\boldsymbol{\mu} = \mathbf{0}$ and $\Sigma = I$. The moments and the Laguerre polynomial approximant of the density function of U are then obtained from Steps 3, 4 and 5.

We make use of a data set that is provided in Hildreth and Lu (1960, p. 58). In this case, there are $k = 5$ independent variables, $p = 18$, the observed value of D is 0.96, and the 13 non-zero eigenvalues of $M(A^*M - t_0 I)$ are those of MA^*M minus t_0. The non-zero eigenvalues of MA^*M are $3.92807, 3.82025$, 3.68089, 3.38335, 3.22043, 2.95724, 2.35303, 2.25696, 1.79483, 1.48804, 0.948635, 0.742294 and 0.378736. For instance, when $t_0 = 1.80977$, which corresponds to the 10^{th} percentile of the simulated cumulative distribution function resulting from 1,000,000 replications, the eigenvalues of the positive definite quadratic form Q_1 are 2.11817, 2.01035, 1.87099, 1.57345, 1.41053, 1.14734, 0.54313 and 0.44706, while those of Q_2 are 0.01507, 0.32186, 0.861265, 1.06761 and 1.43116. The approximate cumulative distribution function of D based on ten moments was evaluated from Equations (3.17) and (3.18) at certain percentiles of the distribution obtained by simulation. The results reported in Table 3 indicate that the empirical and approximate distribution functions are in close agreement for the given simulated percentiles.

Table 3: Approximate CDF evaluated at certain empirical percentile of D.

CDF	Simulated	Approximate CDF
0.01	1.36069	0.010435
0.025	1.51197	0.025476
0.05	1.64792	0.050280
0.1	1.80977	0.099761
0.25	2.08536	0.247875
0.5	2.39014	0.495934
0.75	2.6861	0.748343
0.9	2.93742	0.902156
0.95	3.07679	0.952783
0.975	3.18896	0.977276
0.99	3.31005	0.991466
1	3.83768	1

6. COMPUTATIONAL CONSIDERATIONS AND CONCLUDING REMARKS

Laguerre polynomials for which an explicit representation is provided in this paper are readily available from numerous mathematical packages including *Mathematica* and *Maple*. It should be pointed out that, after the determination of the parameters ν and β, the only remaining step for obtaining a density approximant is the evaluation of the polynomial coefficients, which are easily determined from Equation (3.8). Quantiles can then be obtained by numerical integration or from the explicit representation of the cumulative distribution function given in Equations (3.17) and (3.18) for indefinite quadratic forms. Conveniently, the requisite calculations can be handled by most mathematical or statistical packages. The symbolic computational package *Mathematica* was used for evaluating the approximants and plotting the graphs, the code being available from the authors upon request.

The proposed density approximation methodology is conceptually simple since it is essentially based on a moment-matching technique. Moreover, it is easy to program and consistently yields remarkably accurate percentage points. Although most applications require relatively few moments, the proposed approximation can accommodate a large number of moments, if need be. The applicability of the results is not restricted to quadratic forms since this methodology can also be utilized to approximate the density functions of random variables that are approximately or asymptotically distributed as gamma random variables, such as those that are proportional to the logarithm of the inverse of certain likelihood ratio test statistics or those that can be expressed as general linear combinations of independently distributed non-central chi-square random variables, which occur in asymptotic theory.

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