REVSTAT – Statistical Journal
Volume 12, Number 2, June 2014, 157–168

EXTREMES OF PERTURBED BIVARIATE RAYLEIGH RISKS

Authors: Enkelejd Hashorva
– Department of Actuarial Science, University of Lausanne, Switzerland
enkelejd.hashorva@unil.ch

Saralees Nadarajah
– School of Mathematics, University of Manchester, Manchester M13 9PL, UK
mbbsssn2@manchester.ac.uk

Tibor K. Pogány
– Faculty of Maritime Studies, University of Rijeka, 51000 Rijeka, Croatia
poganj@pfri.hr

Received: January 2013 Revised: April 2013 Accepted: May 2013

Abstract:

- We establish first an asymptotic expansion for the joint survival function of a bivariate Rayleigh distribution, one of the most popular probabilistic models in engineering. Furthermore, we show that the component-wise maxima of a Hüsler–Reiss triangular array scheme of independent perturbed bivariate Rayleigh risks converges to a bivariate Hüsler–Reiss random vector.

Key-Words:

- *asymptotic independence; Gumbel max-domain of attraction; Hüsler–Reiss distribution; Rayleigh distribution; triangular arrays.*

AMS Subject Classification:

- 62E20.

1. INTRODUCTION

Let $(X_1, Y_1), ..., (X_N, Y_N)$ be independent bivariate Gaussian random vectors with $N(0,1)$ distributed marginals and correlation coefficient $\rho \in (-1,1)$. We define a bivariate Rayleigh random vector (risk), (U_m, V_m), by

$$U_m = \sum_{i=1}^{N}\Big(X_i - \sum_{i=1}^{N} X_i/N\Big)^2, \text{ and } V_m = \sum_{i=1}^{N}\Big(Y_i - \sum_{i=1}^{N} Y_i/N\Big)^2,$$

where $m := N-1$. Basic distributional properties of bivariate Rayleigh random vectors are derived in Nadarajah [23]. In view of equation (3) in Nadarajah [23], the joint probability density function (pdf) of (U_m, V_m), $m \geq 1$ is

$$\begin{aligned} h(u,v) \;&=\; \frac{(uv)^{m/2-1}}{\Gamma(m/2)\,(2\widetilde{\varrho})^{m/2}} \exp\Big(-\frac{u+v}{2\widetilde{\varrho}}\Big) \\ &\quad \cdot {}_0F_1\Big(;m/2;\rho^2 uv/\,(2\widetilde{\varrho})^2\Big), \quad \forall u,v \in (0,\infty), \end{aligned} \tag{1.1}$$

where $\rho \in (-1,1)$, $\widetilde{\varrho} := 1-\rho^2$ and

$${}_0F_1\,(;a;x) = \sum_{k=0}^{\infty} \frac{1}{(a)_k}\frac{x^k}{k!}$$

denotes a hypergeometric function, where $(e)_k = e(e+1)\cdots(e+k-1)$ denotes the ascending factorial.

The distribution given by the joint pdf (1.1) is known as the *bivariate Rayleigh distribution*. It has received widespread applications especially in engineering. Some recent applications have included: statistics of wave groups measured in the northern North Sea (Stansell *et al.* [27]); performance analysis of system with selection combining over correlated Rician fading channels in the presence of cochannel interference (Panajotović *et al.* [24]); cochannel interference effect on bit error probability performance of switch and stay combining receiver in correlated Rician fading (Panajotović *et al.* [25]).

The bivariate Rayleigh distribution has also been used to model extreme values, for example, with respect to depth-limited extreme wave heights in a sea state (Méndez and Castanedo [21]), reliability assessment of marine structures (Leira and Myrhaug [17], Leira *et al.* [18]), and asymptotic capacity analysis in point-to-multipoint cognitive radio networks (Ji and Chen [14]). But the asymptotic distribution of the extreme values of (U_m, V_m) has not been known. The principal aim of this short note is to establish the limiting max-stable distribution of (U_m, V_m).

An important max-stable multivariate distribution related to our results is the Hüsler–Reiss distribution due to Hüsler and Reiss [13]. In a bivariate setting, Hüsler–Reiss distribution has the joint cumulative distribution function (cdf)

$$H_\lambda(x,y) = \exp\left[-\Phi\left(\lambda + \frac{x-y}{2\lambda}\right)\exp(-y) \right.$$

$$\left. -\Phi\left(\lambda + \frac{y-x}{2\lambda}\right)\exp(-x)\right], \quad x, y \in \mathbb{R}, \tag{1.2}$$

where $\Phi(\cdot)$ denotes the standard normal cdf and $\lambda \in (0,\infty)$ is a parameter. For any λ, the marginal cdf's of H_λ are the Gumbel cdf's $\Lambda(x) = \exp\{-\exp(-x)\}$, $x \in \mathbb{R}$.

The parameter λ has a nice representation and comes naturally in the setup of Gaussian triangular arrays. Roughly speaking, if $\rho_n \in (-1,1)$ is the correlation coefficient of a bivariate triangular array, then under the Hüsler–Reiss condition

$$\lim_{n\to\infty} (1-\rho_n)\ln n = \lambda^2 \in (0,\infty),$$

the cdf H_λ appears as the limiting distribution of the normalized maxima.

Hüsler–Reiss distribution has received widespread applications. Hüsler–Reiss distribution arises not only as the limiting max-stable distribution of the componentwise maxima of Gaussian random vectors, but it arises also as the limiting max-stable distribution of the componentwise maxima of random vectors having chi-square, elliptically symmetric and other distributions, see Hashorva [10], Frick and Reiss [8] and Hashorva *et al.* [11].

Some applications of Hüsler–Reiss distribution have included: models for environmental data (Joe [15]); portfolio risk measurement (Bouyé [2]); extremal dependence of multivariate catastrophic losses (Lescourret and Robert [19], Haug *et al.* [12]); inference for bivariate survival data (Ding and Wang [4]); models for spatial extremes (Smith and Stephenson [26]); spatial extreme fields (Bacro *et al.* [1]); models for extremes observed in space and time (Davis *et al.* [3]); multivariate value at risks for operational risk capital computation (Guegan and Hassani [9]); extremal discriminant analysis (Manjunath *et al.* [20]); multiasset derivatives and joint distributions of asset prices (Molchanov and Schmutz [22]). Important recent contributions and insights concerning the Hüsler–Reiss distribution can be found in Kabluchko [16] and Engelke *et al.* [5, 6, 7].

It follows from (1.1) that both U_m and V_m are chi-squared random variables with m degrees of freedom. Let G_m denote the cdf's of U_m and V_m. They belong to the Gumbel max-domain of attraction with scaling function $w(t) = 1/2$, i.e.,

$$\lim_{x\to\infty} \frac{1 - G_m\left(x + s/w(x)\right)}{1 - G_m(x)} = \exp(-s), \quad s \in \mathbb{R}.$$

Equivalently,

$$\lim_{n\to\infty} \sup_{s\in\mathbb{R}} \Big| \left(G_m\left(a_n s + b_n\right)\right)^n - \Lambda(s) \Big| = 0$$

with constants

$$a_n = 2, \quad b_n = 2\ln n + (m-2)\ln\ln n - 2\ln\Gamma\left(\frac{m}{2}\right), \quad n > 1.$$

As in Hüsler and Reiss [13] we shall consider a triangular array setup, which is of interest when the components are asymptotically independent. In the Gaussian framework, the asymptotic independence of the components is well known, i.e.,

$$\lim_{u\to\infty} \frac{\mathbb{P}\left(X_1 > u, Y_1 > u\right)}{\mathbb{P}\left(X_1 > u\right)} = 0$$

for any $\rho \in (-1,1)$. In view of Hashorva *et al.* [11], U_m and V_m are asymptotically independent for any $\rho \in (-1,1)$. So, we have

$$\lim_{n\to\infty} n\mathbb{P}\left(U_m > x + b_n, V_m > y + b_n\right) = 0. \tag{1.3}$$

Our first result below presented in Theorem 2.1 gives the exact rate of convergence to zero claimed in (1.3). In the light of the aforementioned paper, the component-wise maxima of bivariate triangular arrays of Rayleigh risks is attracted to the Hüsler–Reiss distribution. Indeed, in order to see that let (U'_m, V'_m) be another bivariate random vector defined by the stochastic representation

$$U'_m = \sum_{i=1}^{m} X_i^2, \quad V'_m = \sum_{i=1}^{m} Y_i^2$$

and further

$$U_m + N\left(\overline{X}_N\right)^2 = U'_{m+1}, \quad V_m + N\left(\overline{Y}_N\right)^2 = V'_{m+1}, \quad N = m+1.$$

Moreover, (U_m, V_m) is independent of $(\overline{X}_N, \overline{Y}_N)$, and $(\sqrt{N}\overline{X}_N, \sqrt{N}\overline{Y}_N)$ has the same distribution as (X_1, Y_1), implying the equality in distribution

$$(U_m, V_m) \stackrel{d}{=} \left(U'_m, V'_m\right).$$

Consequently, in view of Hashorva *et al.* [11] the asymptotic behavior of the component-wise maxima of a Hüsler–Reiss triangular array scheme of bivariate Rayleigh risks is known.

In Section 2, we establish the rate of convergence to zero for the joint survival function $\mathbb{P}\left(U_m > x + b, V_m > y + b\right)$ as b tends to infinity. Then we consider a perturbation of Rayleigh risks and derive the limiting distribution of bivariate maxima of triangular arrays of such risks, which turns out to be the bivariate Hüsler–Reiss distribution. All of the proofs are provided in Section 3.

2. MAIN RESULTS

Our first result derives the exact tail asymptotic behavior of the joint survival function of two bivariate Rayleigh risks, which in particular implies (1.3).

Theorem 2.1. With the notation as in Section 1, for any x, y reals and $\rho \in (-1,1)$, we have

$$\mathbb{P}\left(U_m > x+b, V_m > y+b\right) = \frac{\sqrt{2}|\rho|^{(1-m)/2}\tilde{\varrho}^{3/2}}{\sqrt{\pi}} b^{(m-3)/2} \exp\left(-\frac{b}{1+|\rho|}\right) \cdot \left[1+O\left(b^{-1}\right)\right]$$

as $b \to \infty$.

A direct consequence of Theorem 2.1 is that U_m and V_m are asymptotically independent for any $\rho \in (-1,1)$.

Our second result is concerned with perturbed Rayleigh risks: in order to motivate the definition of such risks, recall that we can write

$$Y_i \stackrel{d}{=} \rho X_i + \sqrt{1-\rho^2} Z_i, \quad 1 \le i \le N$$

with $X_i, Z_i, i \le N$ independent $N(0,1)$ risks. Since in the triangular array framework introduced in Hüsler and Reiss [13] the correlation coefficient $\rho = \rho_n$ tends to one as $n \to \infty$, we see that the base risk is X_i, $i \le N$ and Z_i plays the role of a perturbation. Since as mentioned in Section 1, the case of Rayleigh risks is already dealt with in Hashorva *et al.* [11], we consider the asymptotic distribution of component-wise maxima for triangular arrays of perturbed independent Rayleigh risks. Therefore, we introduce next (X_i, Y_i), $i \ge 1$ with the stochastic representation

$$(X_i, Y_i) \stackrel{d}{=} \left(X, \rho X + \sqrt{1-\rho^2} Z\right), \tag{2.1}$$

where X is a base random variable independent of $Z \sim N(0,1)$. Clearly, if X is also a $N(0,1)$ random variable, then (X_i, Y_i) is a bivariate Rayleigh risk and ρ is the correlation coefficient. Let now $(\mathcal{U}_{m,i}^{(n)}, \mathcal{V}_{m,i}^{(n)})$, $1 \le i \le n$ be independent bivariate random vectors with joint cdf F_{mn} that coincides with the joint cdf of (U_m, V_m) for some fixed $\rho_n \in (-1,1)$, where for the definition of (U_m, V_m) we consider the general bivariate random vectors (X_i, Y_i) given by (2.1) with ρ substituted by $\rho_n \in (-1,1)$. Note that the cdf of V_m depends on n since we use now ρ_n. However, the cdf of U_m does not depend on n.

Under some restrictions on the marginal distributions $F_{mn,i}$, $i = 1, 2$ of F_{mn} we have the following result.

Theorem 2.2. Suppose that for some positive constants $a_n > 0$, $b_n \in \mathbb{R}$, $n \geq 1$ we have

$$\lim_{n\to\infty} \sup_{x\in\mathbb{R}} \Big| (F_{mn,i}(a_n x + b_n))^n - \Lambda(x) \Big| = 0, \quad i = 1, 2.$$

If further the Hüsler–Reiss condition

$$\lim_{n\to\infty} (1-\rho_n)\frac{b_n}{a_n} = \lambda^2 \in [0,\infty) \tag{2.2}$$

holds, then

$$\lim_{n\to\infty} \sup_{x,y\in\mathbb{R}} \Big| (F_{mn}(a_n x + b_n, a_n y + b_n))^n - H_\lambda(x,y) \Big| = 0, \tag{2.3}$$

where H_λ is given in (1.2).

Remarks:

a) The convergence in (2.3) can be stated equivalently as the joint weak convergence of $\left(\max_{i\leq n} \mathcal{U}_{mi}^{(n)}, \max_{i\leq n} \mathcal{V}_{mi}^{(n)}\right)$ as $n \to \infty$.

b) In the case $m = 2$ and the base risk is $X = WI$ with W having $N(0,1)$ distribution and I being a Bernoulli random variable independent of W, we can check that the assumptions of Theorem 2.2 are fulfilled.

3. PROOFS

Proof of Theorem 2.1: Using the well-known fact that

$${}_0F_1(;b,z) \sim \frac{\Gamma(b)}{2\sqrt{\pi}} z^{(1-2b)/4} \exp\left(2\sqrt{z}\right)$$

as $z \to \infty$, we have

$$\begin{aligned}
&\mathbb{P}(U_m > x+b, V_m > y+b)\\
&= \frac{1}{\Gamma(m/2)\,(2\widetilde{\varrho})^{m/2}} \int_{x+b}^{\infty}\int_{y+b}^{\infty} (uv)^{m/2-1} \exp\Big(-\frac{u+v}{2\widetilde{\varrho}}\Big)\\
&\quad \cdot {}_0F_1\left(;m/2;\rho^2 uv/(2\widetilde{\varrho})^2\right) dvdu\\
&\sim \frac{|\rho|^{(1-m)/2}}{2\sqrt{\pi}\,(2\widetilde{\varrho})^{1/2}} \int_{x+b}^{\infty}\int_{y+b}^{\infty} (uv)^{(m-3)/4} \exp\Big(\frac{|\rho|\sqrt{uv}}{\widetilde{\varrho}} - \frac{u+v}{2\widetilde{\varrho}}\Big) dvdu\\
&=: \frac{|\rho|^{(1-m)/2}}{2\sqrt{\pi}\,(2\widetilde{\varrho})^{1/2}} I(b)\,. \qquad (3.1)
\end{aligned}$$

Since $\sqrt{uv} \le (u+v)/2$,

$$\begin{aligned} I(b) &\le \int_{x+b}^{\infty}\int_{y+b}^{\infty}(uv)^{(m-3)/4}\exp\left(-\frac{u+v}{2(1+|\rho|)}\right)dvdu \\ &= \int_{x+b}^{\infty}u^{(m-3)/4}\exp\left(-\frac{u}{2(1+|\rho|)}\right)du\int_{y+b}^{\infty}v^{(m-3)/4}\exp\left(-\frac{v}{2(1+|\rho|)}\right)dv \\ &= \frac{1}{\left[2\left(1+|\rho|\right)\right]^{(m+1)/2}}\Gamma\Big(\frac{m+1}{4},\frac{2x+b}{2(1+|\rho|)}\Big)\Gamma\Big(\frac{m+1}{4},\frac{2y+b}{2(1+|\rho|)}\Big), \end{aligned}$$

where $\Gamma(s,z)=\int_z^\infty t^{s-1}\exp(-t)dt$ denotes the complementary incomplete gamma function. Since

$$\int_z^\infty t^{s-1}\exp(-At)dt = \frac{\Gamma(s,Az)}{A^s}, \qquad A>0$$

and

$$\Gamma(s,z) = \exp(-z)z^{s-1}\left(1+O\left(z^{-1}\right)\right)$$

as $|z|\to\infty$, we conclude that for $|\rho|<1$

$$\begin{aligned} I(b) &= \left[\exp\left(-\frac{x+b/2}{1+|\rho|}\right)\left(\frac{x+b/2}{1+|\rho|}\right)^{\frac{m-3}{4}}\right] \\ &\quad\cdot\left[\exp\left(-\frac{y+b/2}{1+|\rho|}\right)\left(\frac{y+b/2}{1+|\rho|}\right)^{\frac{m-3}{4}}\right]\left(1+O\left(b^{-1}\right)\right) \\ &= b^{\frac{m-3}{2}\exp\left(-\frac{b}{1+|\rho|}\right)\left(1+O\left(b^{-1}\right)\right)} \end{aligned} \tag{3.2}$$

as $b\to\infty$. The proof follows by combining (3.2) and (3.1). □

Proof of Theorem 2.2: Let (U_m, V_{mn}) be a bivariate random vector with the joint cdf F_{mn}. By the assumptions on the marginal distributions of F_{mn}, the proof follows if we show that

$$\begin{aligned} \lim_{n\to\infty} n\mathbb{P}\left(U_m > a_n x + b_n, V_{mn} > a_n y + b_n\right) &= \exp(-x)+\exp(-y)-\ln H_\lambda(x,y) \\ &=: g(x,y) \end{aligned}$$

holds for any $x,y\in\mathbb{R}$. Let $Z, Z_1, ..., Z_n$ be independent $N(0,1)$ random variables and let

$$(X_i, Y_i) \stackrel{d}{=} \left(X_i, \rho_n X_i + \sqrt{1-\rho_n^2}Z_i\right),$$

assuming further that $X_i, Z_i, i\le n$ are mutually independent and $X_i \stackrel{d}{=} X, i\ge 1$. Hence, we obtain

$$\begin{aligned} V_{mn}^2 &\stackrel{d}{=} \sum_{i=1}^N\left(Y_i - \sum_{i=1}^N Y_i/N\right)^2 \\ &\stackrel{d}{=} \sum_{i=1}^N\left(\rho_n\left(X_i-\overline{X}_N\right)-\sqrt{1-\rho_n^2}\left(Z_i-\overline{Z}_N\right)\right)^2 \\ &\stackrel{d}{=} \rho_n^2 U_m - 2\rho_n\sqrt{1-\rho_n^2}T_m + \left(1-\rho_n^2\right)V_m^*, \end{aligned}$$

where

$$T_m=\sum_{i=1}^{N}\left(X_i-\overline{X}_N\right)Z_i,\quad V_m^*=\sum_{i=1}^{N}\left(\overline{Z}_N-Z_i\right)^2.$$

By the independence of $X_i, Z_i, i\le n, \overline{X}_N$ and the fact that $Z, Z_1, ..., Z_N$ are independent $N(0,1)$ random variables, we may further write

$$T_m=\sum_{i=1}^{N}\left(X_i-\overline{X}_N\right)Z_i\overset{d}{=}Z_1\sqrt{\sum_{i=1}^{N}\left(X_i-\overline{X}_N\right)^2}\overset{d}{=}Z\sqrt{U_m}. \tag{3.3}$$

Hence, as in Hashorva *et al.* [11], we have for any $\varepsilon>0$ and any $x,y\in\mathbb{R}$

$$\begin{aligned}
&\mathbb{P}\left(U_m>a_nx+b_n, V_{mn}>a_ny+b_n\right)\\
&\quad=\mathbb{P}\left(U_m>a_nx+b_n,\ \rho_n^2U_m-2\rho_n\sqrt{1-\rho_n^2}T_m+\left(1-\rho_n^2\right)V_m^*>a_ny+b_n\right)\\
&\quad\le\mathbb{P}\Big(U_m>a_nx+b_n,\ \rho_n^2U_m-2\rho_n\sqrt{1-\rho_n^2}T_m+\left(1-\rho_n^2\right)V_m^*>a_ny+b_n,\\
&\qquad\left(1-\rho_n^2\right)V_m^*\le\varepsilon\Big)+\ \mathbb{P}\left(U_m>a_nx+b_n,\ \left(1-\rho_n^2\right)V_m^*>\varepsilon\right)\\
&\quad\le\mathbb{P}\left(U_m>a_nx+b_n,\ \rho_n^2U_m-2\rho_n\sqrt{1-\rho_n^2}Z\sqrt{U_m}>a_ny-\varepsilon+b_n\right)\\
&\qquad+\ \mathbb{P}\left(U_m>a_nx+b_n,\ \left(1-\rho_n^2\right)V_m^*>\varepsilon\right).
\end{aligned}$$

By the assumptions, we have

$$\lim_{n\to\infty}n\mathbb{P}\left(U_m>a_nx+b_n\right)=\exp(-x),\quad\forall x\in\mathbb{R}.$$

Consequently, for some ε sufficiently small

$$\begin{aligned}
&\lim_{n\to\infty}n\mathbb{P}\left(U_m>a_nx+b_n,\left(1-\rho_n^2\right)V_m^*>\varepsilon\right)\\
&=\lim_{n\to\infty}n\mathbb{P}\left(U_m>a_nx+b_n\right)\mathbb{P}\left(\left(1-\rho_n^2\right)V_m^*>\varepsilon\right)=0.
\end{aligned}$$

By the fact that V_m^* is non-negative, we have further

$$\begin{aligned}
&\mathbb{P}\left(U_m>a_nx+b_n, V_{mn}>a_ny+b_n\right)\\
&\ge\mathbb{P}\left(U_m>a_nx+b_n,\rho_n^2U_m-2\rho_n\sqrt{1-\rho_n^2}Z\sqrt{U_m}>a_ny+b_n\right).
\end{aligned}$$

We have with H the cdf of U_m (which does not depend on n)

$$n\left[1-H\left(b_n\right)\right]\to1,\quad\overline{H}_n(x):=\frac{1-H\left(a_nx+b_n\right)}{1-H\left(b_n\right)}\to\exp(-x),\quad\forall x\in\mathbb{R}$$

as $n\to\infty$. Furthermore, by condition (2.2) and the fact that $Z\overset{d}{=}-Z$

$$\begin{aligned}
l_n(x,y)&:=\mathbb{P}\left(\rho_n\left(a_nx+b_n\right)-2\rho_n\sqrt{1-\rho_n^2}\sqrt{a_nx+b_n}Z>a_ny+b_n\right)\\
&\to\mathbb{P}\left(4\lambda Z>2y-2x+2\lambda^2\right),\quad n\to\infty
\end{aligned}$$

holds locally uniformly for $x \in \mathbb{R}$. Using a conditional argument as in Hashorva *et al.* [11] and utilizing further (3.3), we obtain

$$\begin{aligned}
& g(x,y) \\
&= \lim_{n\to\infty} n \int_{a_n x+b_n}^{\infty} \mathbb{P}\left(\rho_n^2 U_m - 2\rho_n\sqrt{1-\rho_n^2}Z\sqrt{U_m} > a_n y + b_n \middle| U_m = s\right) dH(s) \\
&= \lim_{n\to\infty} \frac{1}{1-H(b_n)} \int_x^{\infty} \mathbb{P}\Big(\rho_n^2\left(a_n t + b_n\right) - 2\rho_n\sqrt{1-\rho_n^2}Z\sqrt{U_m} > a_n y + b_n \Big| \\
& \quad U_m = a_n t + b_n\Big)\, dH\left(a_n t + b_n\right) \\
&= -\lim_{n\to\infty} \int_x^{\infty} \mathbb{P}\left(\rho_n^2\left(a_n t + b_n\right) - 2\rho_n\sqrt{1-\rho_n^2}\sqrt{a_n t + b_n}Z > a_n y + b_n\right) dH_n(t) \\
&= -\lim_{n\to\infty} \int_x^{\infty} l_n(t,y)\, dH_n(t) \\
&= \int_x^{\infty} \mathbb{P}\left(Z > (y-t)/(2\lambda) + \lambda/2\right) \exp(-t)\, dt.
\end{aligned}$$

Utilizing the explicit expression of $g(x,y)$ derived in Hüsler and Reiss [13] establishes the proof. □

ACKNOWLEDGMENTS

E. Hashorva acknowledges support by the Swiss National Science Foundation Grants 200021-134785 and 200021-1401633/1. The authors would like to thank the Editor and the referee for careful reading and for their comments which improved the paper.

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