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ESTIMATING THE SHAPE PARAMETER OF TOPP–LEONE DISTRIBUTION BASED ON PROGRESSIVE TYPE II CENSORED SAMPLES

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Abstract:

- In this paper, classical and Bayesian point estimations of the Topp–Leone distribution shape parameter are studied when the sample is progressive Type II censored. The maximum likelihood estimator (MLE) of the unknown parameter is proposed. Since the MLE does not have an explicit form, an approximate MLE has been derived. The Bayes estimate and the associated credible interval are also studied. Lindley's method is proposed to approximate the Bayes estimate. The importance sampling technique is also proposed to approximate the Bayes estimate and to construct the associated credible interval. Monte Carlo simulations are performed to compare the performances of the proposed methods, and two data sets have been analyzed for illustrative purposes.

Key-Words:

- *Bayes estimate; credible interval; Lindley's approximation; maximum likelihood estimate; Markov chain Monte Carlo.*

AMS Subject Classification:

- 62N02, 62F15.

1. INTRODUCTION

Topp and Leone [19] introduced a family of distributions with finite support whose cumulative distribution function (cdf) is given by

$$F(x|\theta,\beta) = \begin{cases} 0\,, & x<0 \\ \left(\frac{x}{\beta}\left(2-\frac{x}{\beta}\right)\right)^{\theta}\,, & 0 \le x < \beta \\ 1\,, & x \ge \beta \end{cases}\,, \qquad \theta>0\,, \tag{1.1}$$

and the probability density function (pdf) is given by

$$f(x|\theta,\beta) = \frac{2\theta}{\beta}\left(1-\frac{x}{\beta}\right)\left(\frac{x}{\beta}\left(2-\frac{x}{\beta}\right)\right)^{\theta-1}, \qquad 0<x<\beta\,,\;\; \theta>0\,. \tag{1.2}$$

For simplicity, we denote this distribution by $TL\,(\theta,\beta)$. Topp–Leone (T-L) distribution is a continuous unimodal distribution with bounded support; this makes it appropriate for modeling lifetime of distributions with finite support. Topp and Leone [19] did not provide any motivation for this family of distributions except to saying that it could be used to model failure data. Nadarajah and Kotz [15] showed that this distribution exhibit bathtub failure rate functions with widespread applications in reliability. Moreover, Ghitany *et al.* [10] showed that T-L distribution possesses some attractive reliability properties such as the bathtub-shape hazard rate, decreasing reversed hazard rate, upside-down mean residual life, and increasing expected inactivity time. Moments for T-L distribution were derived by Nadarajah and Kotz [15]. Zghoul [21] provided expressions for moments of ordered statistics from T-L distribution. Recently, Bayoud [6] derived admissible minimax estimates for the shape parameter of the T-L distribution under squared and linear-exponential loss functions. A reflected version of the Generalized T-L distribution was used by Van Drop and Kotz [20] to fit the U.S. income data for the year 2001 for Caucasian, Hispanic and Afro American populations.

Classical and Bayesian inferences of the parameters of T-L distribution have not yet been studied in the presence of censored samples. In this paper, we study classical and Bayesian estimations for the shape parameter of the T-L distribution when the sample is progressive Type II censored. A Type II progressive censoring scheme can be expressed as follows: suppose that n units are placed on a life test at time zero and the experimenter decides beforehand the quantity m, the number of failures to be observed. When the first failure time $X_{1:m:n}$ is observed, R_1 of the remaining $n-1$ surviving units are randomly selected and removed. At the second observed failure time $X_{2:m:n}$, R_2 of the remaining $n-R_1-2$ surviving units are randomly selected and removed. This experiment terminates at the time $X_{m:m:n}$ when the m^{th} failure is observed, and the remaining $R_m = n - R_1 - R_2 - \cdots - R_{m-1} - m$ surviving units are all removed. The sample $\{X_{1:m:n}, X_{2:m:n}, ..., X_{m:m:n}\}$ is called progressively Type II censored sample of size m from a sample of size n with censoring scheme $\{R_1, R_2, ..., R_m\}$. The values $\{m; R_1, R_2, ..., R_m\}$ are determined prior to the study. Note that, if $R_1 = R_2 = \cdots = R_m = 0$, so that $n = m$, then the progressively Type II censoring scheme reduces to the case of complete sample. Also note that if $R_1 = R_2 = \cdots = R_{m-1} = 0$, so that $R_m = n - m$, then the censoring scheme reduces to a conventional Type II censoring scheme. Readers may refer to [2] for more details about the progressive censoring.

The rest of this paper is organized as follows. In Section 2, we provide the model assumptions based on the progressive Type II censoring. The MLE is studied in Section 3. We propose an approximate MLE (AMLE) in Section 4. The Bayes estimate and the construction of the credible interval are discussed in Section 5. In Section 6, data analysis and some simulation studies are carried out to investigate the performance of the proposed estimation methods. Finally, we conclude the paper in Section 7.

2. MODEL ASSUMPTIONS

Let $X_{1:m:n}, X_{2:m:n}, ..., X_{m:m:n}$ be a progressively Type II censored sample from T-L lifetime distribution (1.2), with $\{m; R_1, R_2, ..., R_m\}$ being the progressive censoring scheme. The likelihood function based on the observed progressive Type II censored sample $D = \{x_{1:m:n}, x_{2:m:n}, ..., x_{m:m:n}\}$ is given by:

$$L(D|\theta,\beta) = c\left(\frac{2\theta}{\beta}\right)^m \prod_{i=1}^{m}\left(1-\frac{x_{i:m:n}}{\beta}\right) u^{(\theta-1)}\left(x_{i:m:n}\right)\left[1-u^{\theta}\left(x_{i:m:n}\right)\right]^{R_i}, \tag{2.1}$$

where

$$c = n(n-1-R_1)(n-2-R_1-R_2)\cdots\left(n-\sum_{i=1}^{m-1}(R_i+1)\right), \qquad 0 < x_{i:m:n} < \beta\,,$$

and

$$u\left(x_{i:m:n}\right) = \frac{x_{i:m:n}}{\beta}\left(2-\frac{x_{i:m:n}}{\beta}\right) \in (0,1) \qquad \forall i = 1,2,...,m\,.$$

The log-likelihood function, $l(D|\theta,\beta) = \ln L(D|\theta,\beta)$, may be written from (2.1) as:

$$l(D|\theta,\beta) \propto m\ln(\theta) + \sum_{i=1}^{m}\theta\ln u\left(x_{i:m:n}\right) + \sum_{i=1}^{m}R_i\ln\left[1-u^{\theta}\left(x_{i:m:n}\right)\right]. \tag{2.2}$$

3. MAXIMUM LIKELIHOOD ESTIMATE

Equating the partial derivative of the log-likelihood function $l(D|\theta,\beta)$ in (2.2) to zero, we have that:

$$\frac{\partial l(D|\theta,\beta)}{\partial\theta} = \frac{m}{\theta} + \sum_{i=1}^{m} \ln u\left(x_{i:m:n}\right) - \sum_{i=1}^{m} \frac{u^{\theta}\left(x_{i:m:n}\right)}{1-u^{\theta}\left(x_{i:m:n}\right)} \ln u\left(x_{i:m:n}\right) R_i = 0\,. \tag{3.1}$$

The MLE of θ is the solution of the likelihood equation (3.1). Since (3.1) is a non-linear equation, a numerical technique is needed. Newton-Raphson method is proposed to obtain the MLE iteratively. A suitable initial guess for the iterative method will be proposed in the next section. However, numerical results, presented in Section 6, show that the numerical MLE converges to the true parameter quite accurately without showing any problem with convergence.

4. APPROXIMATE MAXIMUM LIKELIHOOD ESTIMATE

The likelihood equation (3.1), as mentioned in the previous section, does not admit explicit solution for the shape parameter. Therefore, we expand the function $g_i(\theta) = \frac{u^{\theta}(x_{i:m:n})}{1-u^{\theta}(x_{i:m:n})}$ in a first-order Taylor series around $v_i = \frac{\ln p_i}{\ln u(x_{i:m:n})}$, where $p_i = 1 - \prod_{j=m-i+1}^{m} \frac{j+\sum_{i=m-j+1}^{m} R_i}{1+j+\sum_{i=m-j+1}^{m} R_i}$ for $i = 1, 2, ..., m$. We may then consider the following approximation:

$$g_i(\theta) \approx \frac{u^{v_i}\left(x_{i:m:n}\right)}{1-u^{v_i}\left(x_{i:m:n}\right)} + (\theta - v_i)\frac{u^{v_i}\left(x_{i:m:n}\right)}{\left[1-u^{v_i}\left(x_{i:m:n}\right)\right]^2} \ln u\left(x_{i:m:n}\right)\,. \tag{4.1}$$

Using the approximation in (4.1), (3.1) is roughly:

$$\begin{aligned} &\frac{m}{\theta} + \sum_{i=1}^{m} \ln u\left(x_{i:m:n}\right) - \\ &- \sum_{i=1}^{m} \frac{u^{v_i}\left(x_{i:m:n}\right)}{1-u^{v_i}\left(x_{i:m:n}\right)} \left[1 + \frac{(\theta - v_i)}{1-u^{v_i}\left(x_{i:m:n}\right)} \ln u\left(x_{i:m:n}\right)\right] \ln u\left(x_{i:m:n}\right) R_i = 0\,. \end{aligned} \tag{4.2}$$

From (4.2), we obtain the AMLE of θ as a solution of the quadratic equation:

$$A\theta^2 + B\theta + m = 0\,,$$

where

$$A = -\sum_{i=1}^{m} \frac{u^{v_i}\left(x_{i:m:n}\right)}{\left[1-u^{v_i}\left(x_{i:m:n}\right)\right]^2} \left[\ln u\left(x_{i:m:n}\right)\right]^2 R_i$$

and

$$B = \sum_{i=1}^{m} \ln u\left(x_{i:m:n}\right)\left[1 - R_i \frac{u^{v_i}\left(x_{i:m:n}\right)}{1 - u^{v_i}\left(x_{i:m:n}\right)}\left[1 - v_i \frac{\ln u\left(x_{i:m:n}\right)}{1 - u^{v_i}\left(x_{i:m:n}\right)}\right]\right].$$

Therefore, the AMLE, say $\hat{\theta}_{AMLE}$, is obtained as

$$\hat{\theta}_{AMLE} = \frac{-B - \sqrt{B^2 - 4Am}}{2A}, \tag{4.3}$$

which is the only positive root. This procedure has been used, for example, by Balakrishnan and Aggarwala [2], Balakrishnan and Varadan [3], Balosooriya and Balakrishnan [4] and Kim and Han [12].

It is worth mentioning that the proposed AMLE (4.3) may provide a convenient starting value for the iterative solution for the MLE in (3.1).

5. BAYESIAN INFERENCE

In this section, we discuss the Bayes estimate and the associated credible interval for the shape parameter. The squared error loss function (SELF) is considered, which is defined as

$$L(\hat{\theta}) = \left(\theta - \hat{\theta}\right)^2,$$

where $\hat{\theta}$ is the estimator of θ.

5.1. Prior and posterior analysis

The shape parameter θ is positive. So, it is assumed that θ has an Exponential prior with pdf:

$$g(\theta) = a e^{-a\theta}, \qquad \theta > 0 \quad \text{and} \quad a > 0.$$

This prior is conjugate when the complete sample is considered; see [6].

It follows, from (2.1) and the prior pdf, that the posterior density function of θ can be written as:

$$\begin{aligned} \pi(\theta|D,\beta) &= \frac{L(D|\theta,\beta) g(\theta)}{\int_0^\infty L(D|\theta,\beta) g(\theta)\, d\theta} \\ &= \frac{\theta^m e^{-a\theta} \prod_{i=1}^m u^\theta\left(x_{i:m:n}\right)\left[1 - u^\theta\left(x_{i:m:n}\right)\right]^{R_i}}{K}, \end{aligned} \tag{5.1}$$

where $K = \int_0^\infty \theta^m e^{-a\theta} \prod_{i=1}^m u^\theta(x_{i:m:n}) \left[1 - u^\theta(x_{i:m:n})\right]^{R_i} d\theta$, the normalizing constant.

Under the SELF, the Bayes estimate of θ, say $\hat{\theta}_B(a)$, is the posterior mean, which is given by:

$$\hat{\theta}_B(a) = E_\pi(\theta|D,\beta) = \frac{1}{K}\int_0^\infty \theta^{m+1} e^{-a\theta} \prod_{i=1}^m u^\theta(x_{i:m:n}) \left[1 - u^\theta(x_{i:m:n})\right]^{R_i}. \tag{5.2}$$

It is obvious that (5.2) cannot be evaluated explicitly. Therefore, we propose two approaches to approximate (5.2): Lindley's procedure and the MCMC using the importance sampling technique.

5.2. Lindley's approximation

Lindley [14] proposed an approximation procedure to evaluate the ratio of two integrals, such that the Bayes estimate in (5.2) takes a form containing no integrals. This procedure has been used by several authors in the literature to obtain the Bayes estimates for various distributions; see, for instance, Press [18]. Consider:

$$I(D,a) = \frac{\int y(\theta) e^{l(\theta)+\tau(\theta)} d\theta}{\int e^{l(\theta)+\tau(\theta)} d\theta} \tag{5.3}$$

where l is the log-likelihood function of the observed sample, $y(\theta)$ is a continuous function in θ, and $\tau(\theta) = \ln g(\theta)$ where $g(\theta)$ is the prior pdf of θ.

Based on Lindley's procedure, the ratio (5.3) is approximated by:

$$I(D,a) \approx y\left(\hat{\theta}\right) + \frac{1}{2}\left(\hat{y}_{\theta\theta} + 2\hat{y}_\theta \hat{\tau}_\theta\right)\hat{\sigma}_{\theta\theta} + \frac{1}{2}\left(\hat{y}_\theta \hat{\sigma}^2_{\theta\theta} \hat{l}_{\theta\theta\theta}\right) \tag{5.4}$$

where $y_{\theta\theta}$ denotes the second derivative of the function $y(\theta)$ with respect to θ, $\hat{y}_{\theta\theta}$ represents the same expression evaluated at $\theta = \hat{\theta}_{MLE}$, $\hat{\tau}_\theta = \frac{\partial}{\partial\theta}\tau(\theta)|_{\theta=\hat{\theta}_{MLE}}$, $\hat{l}_{\theta\theta} = \frac{\partial^2 l}{\partial\theta^2}|_{\theta=\hat{\theta}_{MLE}}$, $\hat{l}_{\theta\theta\theta} = \frac{\partial^3 l}{\partial\theta^3}|_{\theta=\hat{\theta}_{MLE}}$ and $\hat{\sigma}_{\theta\theta} = -\frac{1}{\hat{l}_{\theta\theta}}$.

Hence, the approximate Bayes estimate can be obtained using Lindley's procedure, by substituting $y(\theta) = \theta$, l=log-likelihood function (2.2) and $g(\theta) = ae^{-a\theta}$ in Lindley's approximation (5.4), as:

$$\hat{\theta}_{B,L}(a) \approx \hat{\theta}_{MLE} + \frac{a}{\hat{l}_{\theta\theta}} + \frac{1}{2}\frac{\hat{l}_{\theta\theta\theta}}{\hat{l}^2_{\theta\theta}} \tag{5.5}$$

where $\hat{\theta}_{MLE}$ is the MLE of θ,

$$\hat{l}_{\theta\theta} = \frac{\partial^2 l}{\partial \theta^2}|_{\theta=\hat{\theta}_{MLE}} = -\frac{m}{\hat{\theta}^2_{MLE}} - \sum_{i=1}^{m} [\ln u\,(x_{i:m:n})]^2 \frac{u^{\hat{\theta}_{MLE}}\,(x_{i:m:n})}{\left[1-u^{\hat{\theta}_{MLE}}\,(x_{i:m:n})\right]^2} R_i\,,$$

and

$$\begin{aligned}\hat{l}_{\theta\theta\theta} &= \frac{\partial^3 l}{\partial \theta^3}|_{\theta=\hat{\theta}_{MLE}} \\ &= \frac{2m}{\hat{\theta}^3_{MLE}} - \sum_{i=1}^{m} [\ln u\,(x_{i:m:n})]^3 \frac{u^{\hat{\theta}_{MLE}}\,(x_{i:m:n})\left[1+u^{\hat{\theta}_{MLE}}\,(x_{i:m:n})\right]}{\left[1-u^{\hat{\theta}_{MLE}}\,(x_{i:m:n})\right]^3} R_i\,.\end{aligned}$$

5.3. MCMC method

Unfortunately, Lindley's procedure fails to construct credible intervals for the unknown parameter. Hence, we propose to use the importance sampling technique to approximate the Bayes estimate and to construct the associated credible interval. Similar procedure was used, for example, by Chen *et al.* [7], Kundu and Pradhan [13], Pradhan and Kundu [16] and [17]. To implement the importance sampling technique, we rewrite the posterior pdf (5.1) as follows:

$$\pi(\theta|D,\beta) \propto f_1(\theta|D)\, f_2(\theta)$$

where

$$f_1(\theta|D) = \frac{[a-\sum_{i=1}^m \ln u(x_{i:m:n})]^m}{\Gamma(m+1)}\;\theta^m e^{-\theta[a-\sum_{i=1}^m \ln u(x_{i:m:n})]}$$

which is clearly a gamma density function with the shape parameter $(m+1)$ and scale parameter $[a-\sum_{i=1}^m \ln u(x_{i:m:n})]^{-1}$; and

$$f_2(\theta) = \prod_{i=1}^{m} \left[1-u^{\theta}(x_{i:m:n})\right]^{R_i}.$$

Therefore, (5.2) can be written as:

$$\hat{\theta}_B\,(a) = \frac{\int_0^\infty \theta f_1(\theta|D) f_2(\theta) d\theta}{\int_0^\infty f_1(\theta|D) f_2(\theta) d\theta}\,. \tag{5.6}$$

Now, we propose the following algorithms, along the line of Kundu and Pradhan [13], to compute the approximate Bayes estimate and to construct the associated credible interval for the parameter θ.

5.3.1. Algorithm 1 (BE)

Step 1) Generate a random sample of size M from $f_1(\theta|D)$, gamma density function with the shape parameter $(m+1)$ and scale parameter $[a - \sum_{i=1}^{m} \ln u(x_{i:m:n})]^{-1}$, say $\theta_1, \theta_2, ..., \theta_M$;

Step 2) Compute $f_2(\theta_j) = \prod_{i=1}^{m} [1 - u^{\theta_j}(x_{i:m:n})]^{R_i}$, for $j = 1, 2, ..., M$;

Step 3) Under the assumption of SELF, a simulation consistent estimate of θ can be obtained using the importance sampling technique as:

$$\hat{\theta}_{B,IS}\,(a) \;=\; \frac{\sum_{j=1}^{M} \theta_j f_2(\theta_j)}{\sum_{j=1}^{M} f_2(\theta_j)} \;.$$

Using this algorithm, it is possible to construct the Bayes estimate of any function of θ, say $H(\theta)$ as:

$$\hat{H}(\theta) = \frac{\sum_{j=1}^{M} H(\theta_j) f_2(\theta_j)}{\sum_{j=1}^{M} f_2(\theta_j)}, \quad \text{provided that } \hat{H}(\theta) \text{ is defined at all } j = 1, 2, ..., m\,.$$

Now, to compute the credible interval of θ. Let, for $0 < p < 1$, θ_p be such that $P(\theta \le \theta_p|D, \beta) = \int_0^{\theta_p} \pi(\theta|D, \beta)d\theta = p$, where $\pi(\theta|D, \beta)$ is the posterior pdf defined in (5.1).

5.3.2. Algorithm 2 (credible interval)

Here, we use the sample $\theta_1, \theta_2, ..., \theta_M$ that is obtained from Algorithm 1.

Step 1) Compute $w_j = \frac{f_2(\theta_j)}{\sum_{j=1}^{M} f_2(\theta_j)}$ for $j = 1, 2, ..., M$;

Step 2) Arrange the set $\{(\theta_1, w_1), (\theta_2, w_2), ..., (\theta_M, w_M)\}$ as

$$\big\{(\theta_{(1)}, w_{[1]}), (\theta_{(2)}, w_{[2]}), ..., (\theta_{(M)}, w_{[M]})\big\}\;,$$

where $\theta_{(1)} \le \theta_{(2)}, ..., \le \theta_{(M)}$;

Step 3) The $100(1-\alpha)\%$ credible interval for θ is given by:

$$\left(\hat{\theta}_{\frac{\alpha}{2}}, \hat{\theta}_{1-\frac{\alpha}{2}}\right)$$

where $\hat{\theta}_p$ is a simulation consistent Bayes estimate for θ_p, which is given by $\theta_{(M_p)}$ such that M_p is the integer satisfying:

$$\sum_{j=1}^{M_p} w_{[j]} \;\le\; p \;<\; \sum_{j=1}^{M_p+1} w_{[j]}\;.$$

Proposition 5.1. *The posterior pdf $\pi(\theta|D,\beta)$ in (5.1) is log-concave.*

Proof: Since $u(x_i) = \frac{x_i}{\beta}(2 - \frac{x_i}{\beta}) > 0$, then it is easy to see that:

$$\frac{\partial^2 \ln \pi(\theta|D,\beta)}{\partial \theta^2} = -\left[\frac{m}{\theta^2} + \sum_{i=1}^{m} \left[\ln u\left(x_{i:m:n}\right)\right]^2 \frac{u^\theta\left(x_{i:m:n}\right)}{\left[1 - u^\theta\left(x_{i:m:n}\right)\right]^2} R_i\right] < 0$$

for any θ, this proves the result. □

Since the posterior distribution (5.1) is log-concave, then one can apply Devroye's algorithm introduced in Devroye [8] to generate a sample from the posterior distribution, say $\theta_1, \theta_2, ..., \theta_M$. Based on this sample and under the SELF, the approximate Bayes estimate of θ is given by:

$$\hat{\theta}_{MCMC} = \hat{E}(\theta|D) = \frac{1}{M}\sum_{j=1}^{M} \theta_j \, .$$

The $100(1-\alpha)\%$ credible interval of θ can be computed by ordering $\theta_1, \theta_2, ..., \theta_M$ as $\theta_{(1)} \leq \theta_{(2)} \leq ... \leq (\theta_M)$ and taking the interval as:

$$\left(\theta_{(M(\frac{\alpha}{2}))}, \, \theta_{(M(1-\frac{\alpha}{2}))}\right) .$$

6. SIMULATION STUDY AND DATA ANALYSIS

6.1. Simulations

In this section, we present some simulation studies to observe the behavior of the proposed estimation methods for different sample sizes, different priors and for different censoring schemes. We have considered three sample sizes, $n = 15$, 25 and 50; and three progressive Type II censoring schemes with $m = 5$, namely, $(n-m, 0, 0, 0, 0)$, $(0, 0, 0, 0, n-m)$ and $(R_1, R_2, R_3, R_4, R_5)$ where $R_i = \frac{n-m}{m}$ for $i = 1, 2, ..., 5$.

In all cases, the parameter β is assumed without loss of generality to equal 1. Simulations are performed for three values of the shape parameter, namely, $\theta = 0.5$, $\theta = 1$ and $\theta = 10$. For a given n, m and $(R_1, R_2, ...R_m)$, we have generated a sample for the given censoring scheme. The AMLE is computed for the shape parameter based on the method proposed in Section 3. We use this AMLE as a starting value to obtain the MLE iteratively by using Newton-Raphson method as discussed in Section 2. The approximate Bayes estimate is computed for the shape parameter using Lindley's procedure and the importance sampling technique based on 1000 importance sampling. For Bayesian estimation, the following priors are considered: Prior 0: assuming $a = 0.0001$, a very small value, and Prior 1: informative prior with $a \approx 1/\theta$ and $a \approx 2/\theta$, separately, since $E(\theta) = 1/a$. The expected value and the corresponding mean squared error (MSE) of the proposed estimates are computed over 1000 replications. The results are reported in Tables 1, 2 and 3 when $\theta = 0.5, \theta = 1$ and $\theta = 10$, respectively.

From Tables 1, 2 and 3, it is clear that as the sample size increases, the MSE decreases for all estimation methods. This verifies the consistency of the proposed methods. It is also obvious that the AMLE and the approximate Bayes estimates under Prior 1 perform, in terms of MSE, better than the iterative MLE and the approximate Bayes estimates under Prior 0. For fixed sample size n, fixed θ and for any censoring scheme, the approximate Bayes estimates under Prior 1 with $a \approx 2/\theta$ outperform the other estimates in terms of the MSE. It is noticeable that the AMLE performs better than the MLE in all cases. The approximate Bayes estimates under Prior 0 do not perform as efficiently as the other estimates.

Table 1: Expected value of the proposed estimators and the corresponding MSE when $\theta = 0.5$.

n	**Scheme**	$\hat{\theta}_{MLE}$	$\hat{\theta}_{AMLE}$	$\hat{\theta}_{B,L}(a)$			$\hat{\theta}_{B,IS}(a)$		
				$a=2$	$a=4$	$a=10^{-4}$	$a=2$	$a=4$	$a=10^{-4}$
15	(0,0,0,0,10)	0.532 0.0204	0.521 0.0193	0.530 0.0170	0.483 0.0112	0.587 0.0353	0.528 0.0175	0.488 0.0132	0.586 0.0356
	(2,2,2,2,2)	0.534 0.0235	0.477 0.0167	0.534 0.0192	0.481 0.0120	0.593 0.0394	0.533 0.0200	0.488 0.0141	0.593 0.0392
	(10,0,0,0,0)	0.533 0.0308	0.517 0.0286	0.543 0.0252	0.472 0.0125	0.623 0.0604	0.540 0.0254	0.483 0.0166	0.624 0.0607
25	(0,0,0,0,20)	0.522 0.0145	0.516 0.0141	0.524 0.0131	0.497 0.0105	0.555 0.0190	0.515 0.0137	0.490 0.0123	0.548 0.0188
	(4,4,4,4,4)	0.523 0.0167	0.474 0.0128	0.527 0.0150	0.493 0.0111	0.562 0.0226	0.524 0.0155	0.493 0.0122	0.558 0.0225
	(20,0,0,0,0)	0.524 0.0252	0.509 0.0232	0.541 0.0230	0.494 0.0131	0.587 0.0360	0.537 0.0225	0.497 0.0153	0.588 0.0364
50	(0,0,0,0,45)	0.509 0.0078	0.507 0.0077	0.514 0.0075	0.500 0.0063	0.532 0.0102	0.476 0.0093	0.452 0.0097	0.505 0.0117
	(9,9,9,9,9)	0.508 0.0087	0.470 0.0076	0.515 0.0084	0.498 0.0075	0.534 0.0111	0.494 0.0099	0.474 0.0094	0.520 0.0114
	(45,0,0,0,0)	0.503 0.0135	0.491 0.0133	0.527 0.0138	0.497 0.0114	0.561 0.0215	0.522 0.0133	0.495 0.0120	0.563 0.0221

Table 2: Expected value of the proposed estimators and the corresponding MSE when $\theta = 1$.

n	**Scheme**	$\hat{\theta}_{MLE}$	$\hat{\theta}_{AMLE}$	$\hat{\theta}_{B,L}(a)$			$\hat{\theta}_{B,IS}(a)$		
				$a=1$	$a=2$	$a=10^{-4}$	$a=1$	$a=2$	$a=10^{-4}$
15	(0,0,0,0,10)	1.08 0.1010	1.06 0.0946	1.08 0.0816	0.978 0.0524	1.13 0.1231	1.07 0.0842	0.989 0.0632	1.13 0.1228
	(2,2,2,2,2)	1.08 0.1060	0.969 0.0760	1.08 0.0853	0.974 0.0513	1.15 0.1427	1.08 0.0884	0.990 0.0628	1.15 0.1434
	(10,0,0,0,0)	1.09 0.1500	1.06 0.1370	1.11 0.1160	0.956 0.0509	1.19 0.2014	1.10 0.1190	0.981 0.0684	1.20 0.2032
25	(0,0,0,0,20)	1.03 0.0611	1.02 0.0594	1.04 0.0549	0.994 0.0443	1.09 0.0709	1.02 0.0559	0.973 0.0504	1.08 0.0751
	(4,4,4,4,4)	1.03 0.0694	0.937 0.0548	1.04 0.0619	0.985 0.0465	1.10 0.0792	1.04 0.0648	0.981 0.0507	1.10 0.0784
	(20,0,0,0,0)	1.04 0.1020	1.01 0.0960	1.08 0.0925	0.986 0.0527	1.16 0.1368	1.07 0.0917	0.993 0.0612	1.16 0.1388
50	(0,0,0,0,45)	1.02 0.0352	1.05 0.0347	1.03 0.0340	0.996 0.0245	1.06 0.0356	0.953 0.0384	0.903 0.0372	0.990 0.0390
	(9,9,9,9,9)	1.02 0.0387	0.947 0.0326	1.04 0.0374	0.991 0.0260	1.07 0.0466	1.00 0.0431	0.947 0.0352	1.04 0.0470
	(45,0,0,0,0)	1.01 0.0628	0.992 0.0600	1.06 0.0639	0.983 0.0408	1.12 0.0920	1.05 0.0618	0.979 0.0427	1.13 0.0930

Table 3: Expected value of the proposed estimators and the corresponding MSE when $\theta = 10$.

n	**Scheme**	$\hat{\theta}_{MLE}$	$\hat{\theta}_{AMLE}$	$\hat{\theta}_{B,L}(a)$			$\hat{\theta}_{B,IS}(a)$		
				$a=0.1$	$a=0.2$	$a=10^{-4}$	$a=0.1$	$a=0.2$	$a=10^{-4}$
15	(0,0,0,0,10)	10.7 11.1	10.5 10.6	10.5 8.9	9.6 4.9	11.5 13.0	10.6 9.4	9.8 5.9	11.4 12.8
	(2,2,2,2,2)	10.8 12.2	9.6 8.9	10.7 9.7	9.6 5.1	11.7 14.7	10.7 9.9	9.7 6.1	11.7 14.7
	(10,0,0,0,0)	10.8 15.0	10.5 13.4	11.0 11.5	9.4 5.2	12.2 21.3	10.9 11.9	9.6 6.9	12.3 21.5
25	(0,0,0,0,20)	10.5 5.8	10.3 5.6	10.5 5.2	10.0 4.1	10.9 7.4	10.3 5.5	9.8 4.8	10.8 7.6
	(4,4,4,4,4)	10.5 6.5	9.5 4.9	10.6 5.9	10.0 4.5	11.1 8.6	10.5 6.1	10.0 5.0	11.0 8.7
	(20,0,0,0,0)	10.4 9.4	10.1 8.9	10.7 8.6	9.8 5.3	11.8 14.9	10.7 8.4	9.9 6.1	11.8 15.1
50	(0,0,0,0,45)	10.1 3.1	10.1 3.1	10.2 3.0	10.1 2.7	10.6 3.9	9.5 3.8	9.2 3.9	9.9 4.2
	(9,9,9,9,9)	10.2 3.8	9.4 3.2	10.3 3.7	10.1 3.0	10.6 4.5	9.9 4.3	9.6 3.8	10.3 4.8
	(45,0,0,0,0)	10.1 6.3	9.9 6.0	10.6 6.5	10.0 4.4	11.2 10.2	10.5 6.2	10.0 4.7	11.3 10.4

6.2. Data analysis

In this section, we analyze real and simulated data sets using the proposed estimation methods for illustrative purposes.

6.2.1. Real data

We analyze the failure time (in mileage) of eighteen military carriers presented by Grubbs [11] as follows:

$$162,\ 200,\ 271,\ 302,\ 393,\ 508,\ 539,\ 629,\ 706,\ 777,$$
$$884,\ 1101,\ 1182,\ 1463,\ 1603,\ 1984,\ 2355,\ 2880\,.$$

First, it was checked whether the T-L distribution can be used or cannot to analyze this data set. The MLE of β is 2880, the maximum order statistic, and the MLE of θ is 1.133 . The Bayes estimate of θ, under the SELF, is 1.125 when $a = 1$, see [6]. It is obvious that the MLE and the Bayes estimate are almost the same. The Kolmogorov-Smirnov (KS) distance between the empirical distribution function and the fitted distribution function, using the MLEs, has been used to check the goodness of fit. The KS statistic value is 0.135, and the KS critical value is 0.309 at at $n = 18$ and $\alpha = 0.05$. Accordingly, one cannot reject the hypothesis that the data are coming from T-L distribution. We consider the following censoring schemes, assuming $m = 6$:

Scheme 1) $(R_1 = R_2 = \cdots = R_5 = 0, R_6 = 12)$.

Scheme 2) $(R_1 = R_2 = \cdots = R_5 = R_6 = 2)$.

Scheme 3) $(R_1 = 12, R_2 = \cdots = R_5 = R_6 = 0)$.

Based on Schemes 1, 2 and 3, we have generated the following progressive Type II censored samples:

$$D = (162, 200, 271, 302, 393, 508)\ ,$$
$$D = (162, 271, 393, 508, 539, 884)$$

and

$$D = (162, 302, 508, 777, 884, 1463)\ ,$$

respectively. The proposed estimates and the credible interval for the shape parameter are computed and reported in Table 4. It is observed from Table 4 that all estimates are in quite similar agreement and close to the estimates obtained using the complete sample. The approximate Bayes estimates dominate the other estimates when the hyper-parameter a is assumed to equal 1. The associated credible intervals for the shape parameter are satisfactory in all the cases.

Table 4: Real Data Analysis.

Estimate		Scheme 1	Scheme 2	Scheme 3
		Censoring Scheme		
$\hat{\theta}_{MLE}$		1.169	1.303	1.241
$\hat{\theta}_{AMLE}$		1.153	1.289	1.236
$\hat{\theta}_{B,L}(a)$	$a = 0.5$	1.205	1.346	1.307
	$a = 1$	1.163	1.290	1.246
	$a = 3.5$	0.996	1.066	1.002
$\hat{\theta}_{B,IS}(a)$	$a = 0.5$	1.151	1.307	1.285
	$a = 1$	1.128	1.287	1.246
	$a = 3.5$	1.085	1.141	1.053
90% Credible Interval	$a = 0.5$	$(0.76, 1.51)$	$(0.69, 1.49)$	$(0.79, 1.85)$
	$a = 1$	$(0.74, 1.47)$	$(0.66, 1.47)$	$(0.77, 1.84)$
	$a = 3.5$	$(0.66, 1.38)$	$(0.58, 1.27)$	$(0.65, 1.48)$

6.2.2. Simulated data

We analyze the following simulated data set presented by Genc [9] assuming $\theta = 0.3$ and $\beta = 1$:

$$0.1425, 0.2707, 0.2783, 0.0718, 0.4537, 0.0615, 0.0047, 0.3454, 0.4428, 0.1909,$$

$$0.1028, 0.0013, 0.0592, 0.5413, 0.2442, 0.0001, 0.0002, 0.0178, 0.0114, 0.5388$$

We consider the following censoring schemes, assuming $m = 4$:

Scheme 1) $(R_1 = R_2 = R_3 = 0, R_4 = 16)$.

Scheme 2) $(R_1 = R_2 = R_3 = R_4 = 4)$.

Scheme 3) $(R_1 = 16, R_2 = R_3 = R_4 = 0)$.

Based on Schemes 1, 2 and 3, we have generated the following progressive Type II censored samples:

$$D = (0.0001, 0.0002, 0.0013, 0.0047),$$
$$D = (0.0001, 0.0047, 0.01114, 0.0178)$$

and

$$D = (0.0001, 0.0013, 0.0718, 0.2707)\ , \qquad \text{respectively.}$$

The proposed estimates and the credible interval are computed and reported in Table 5. It is clear from Table 5 that all estimates are quite similar, and the approximate Bayes estimates dominate the other when the hyper-parameter $a = 5$. It is also observed that the credible intervals are satisfactory under all the cases.

Table 5: Simulated Data Analysis.

Estimate		Censoring Scheme		
		Scheme 1	**Scheme 2**	**Scheme 3**
$\hat{\theta}_{MLE}$		0.3699	0.4381	0.3664
$\hat{\theta}_{AMLE}$		0.3694	0.4266	0.3662
$\hat{\theta}_{B,L}(a)$	$a = 0.75$	0.3901	0.4651	0.4041
	$a = 3.5$	0.3674	0.4294	0.3720
	$a = 5$	0.3550	0.4100	0.3546
$\hat{\theta}_{B,IS}(a)$	$a = 0.75$	0.3541	0.4731	0.3972
	$a = 3.5$	0.3792	0.4120	0.3675
	$a = 5$	0.3530	0.4054	0.3555
90% Credible Interval	$a = 0.75$	$(0.25, 0.46)$	$(0.23, 0.46)$	$(0.24, 0.58)$
	$a = 3.5$	$(0.24, 0.47)$	$(0.21, 0.44)$	$(0.22, 0.56)$
	$a = 5$	$(0.23, 0.45)$	$(0.21, 0.42)$	$(0.22, 0.51)$

7. CONCLUSIONS

In this article, classical and Bayesian point estimations were proposed for the shape parameter of the Topp–Leone distribution when the sample is progressive Type II censored. It was observed that the MLE cannot be derived in explicit form. Hence, an approximate MLE was proposed. Bayes estimate of the shape parameter cannot be obtained in explicit form. Lindley's procedure and the importance sampling technique were proposed to obtain the approximate Bayes estimate and to construct the credible interval for the shape parameter. The performance of the different estimation methods was compared by Monte Carlo simulations. It was observed that the approximate Bayes estimates, based on the informative prior with $a \approx 2/\theta$, outperform the other estimates in terms of the MSE. It was also noticed that the AMLE performs well and dominates the MLE in terms of the MSE in all cases.

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