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OPTIMIZING THE SIMPLE STEP STRESS ACCELERATED LIFE TEST WITH TYPE I CENSORED FRÉCHET DATA

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Abstract:

- In this paper, we propose an optimization for the simple step stress accelerated life test for the Fréchet distribution under type I censoring. The extreme value distribution has recently become increasingly important in engineering statistics as a suitable model to represent phenomena with extreme observations. One probability distribution, that is used to model the maximum extreme events, is the Fréchet (extreme value type II) distribution. A log-linear relationship between the Fréchet scale parameter and the stress are assumed. Furthermore, we model the effects of changing stress as a cumulative exposure function. The maximum likelihood estimators of the model parameters are derived. By minimizing the asymptotic variance of the desired life estimate and the reliability estimate, we obtain the optimal simple step stress accelerated life test. Finally, the simulation results are discussed to illustrate the effect of the initial estimates on the optimal values.

Key-Words:

- *Fréchet distribution; log-linear relationship; maximum likelihood estimator; optimal design; reliability; step stress accelerated life test; type I censored data.*

AMS Subject Classification:

- 90B25, 62P30.

1. INTRODUCTION

Nowadays, manufacturers face strong pressure to rapidly develop new, higher technology products, while improving the productivity. This has motivated the development of methods such as concurrent engineering and encouraged wider use of the designed experiments for product and process improvement. The requirements for higher reliability have increased the need for more up front testing of the materials, components, and systems. This is in line with the modern quality philosophy for producing the high reliability products: achieve high reliability by improving the design and manufacturing processes; move away from reliance on inspection (or screening) to achieve high reliability [12].

Estimating the failure time distribution of components including high reliability products is particularly difficult. Most modern products are designed to operate without failure for years, decades, or longer. Thus, a few units will fail or degrade in a test under the normal conditions. For example, the design and construction of a communication satellite, may allow only 8 months to test the components that are expected to be in service for 10 or 15 years. A method for obtaining information on the life distribution of a product in a timely fashion, is to test it on an unusually high level of stress (*e.g.*, high levels of temperature, voltage, pressure, vibration, cycling rate, or load) in order to provoke early failures. These methods are called the accelerated life tests. The results of this test are then used to estimate the life distribution of the product.

Engineers in manufacturing industries have used accelerated life test (ALT) experiments for many decades. The purpose of ALT experiments is to acquire reliable information quickly.

According to Bai *et al.* [4] and Nelson [17], one way of applying stress to the test is a step-stress scheme which allows the stress setting of a unit to be changed at pre-specified times or upon the occurrence of a fixed number of failures. This scheme is called step stress accelerated life test (SSALT), which is considered in this paper.

To implement the SSALT, we first apply a low stress to all products, if a product endures the stress (does not fail) we apply a higher stress, if only one change of the stress level is done, it is called a simple step-stress accelerated life test.The objective of the SSALT experiment is to estimate the percentile life or reliability prediction by choosing the optimal time of increasing the level of stress that leads to the most accurate estimate. Our main objective is to choose the times to change the stress level in such a way that the variance of estimator of above parameters is minimized under a natural stress level.

The step-stress procedure was first introduced, with the cumulative exposure model, by Nelson [1]. Miller and Nelson [13] provided the optimum simple stress plans for the accelerated life testing, where life products are assumed to have exponentially distributed lifetimes, Bai *et al.* [4] extended the results of Miller and Nelson [13] to the case of censoring. Khamis and Higgins [6, 7] obtained the stress change time which minimizes the asymptotic variance of maximum likelihood estimate of the log mean life at the design condition. Alhadeed and Yang [2] discussed the optimal simple step-stress plan for the Khamis–Higgins model. Most of the available literature on step-stress accelerated life testing deals with the exponential, and Weibull distributions.

The extreme value distribution becomes increasingly important in engineering statistics as a suitable model to represent the phenomena with large extreme observations. In engineering, this distribution is often called the Fréchet model. It is one of the pioneers in extreme value statistics. The Fréchet distribution is one of the probability distributions used to model the maxima extreme events. Thus, the Fréchet distribution is well suited to characterize the random variables of large features and components with a high reliability products. Therefore, it is an important distribution for modeling the statistical behavior of material properties for a variety of engineering applications.

Fréchet distribution is a popular model for lifetimes. Some recent applications have involved the modeling of failure times of air-conditioning systems in jet planes [11] and the modeling of the behavior of off-site AC power failure recovery times at three nuclear plant sites [3] Some results for beta Fréchet distribution are given by [5].

In spite of its popularity, Fréchet distribution has not been used as a lifetime distribution in simple step stress accelerated life test analysis. This paper is the first attempt in this regard. We implement the SSALT analysis and design, by assuming that the failure time of test products follows the Fréchet distribution.

The contents of this paper are organized as follows. The model and basic assumptions are presented in section 2. The maximum likelihood estimators (MLEs) and Fisher information matrix are given in section 3. The optimal test design is derived in section 4, which is followed by a simulation study.

2. MODEL AND TEST PROCEDURE

The Fréchet distribution is a special case of the generalized extreme value distribution. The Fréchet distribution has applications ranging from an accelerated life testing through to earthquakes, floods, horse racing, rainfall, queues in supermarkets, sea currents, wind speeds and track race records. Kotz and Nadarajah [8] give some applications in their book.

To develop appropriate probabilistic models and assess the risks caused by these events, business analysts and engineers frequently use extreme value distributions.

The Fréchet distribution was named after the French mathematician Maurice Fréchet (1878–1973). It is also known as the Extreme Value Type II distribution. It has the cumulative distribution function (CDF) specified by

$$F(t) = \exp\left\{-\left(\frac{t}{\theta}\right)^{-\alpha}\right\} \tag{2.1}$$

for $t > 0$, $\alpha > 0$ and $\theta > 0$. The corresponding probability density function (PDF) is

$$f(t) = \frac{\alpha}{\theta}\left(\frac{t}{\theta}\right)^{-\alpha-1}\exp\left\{-\left(\frac{t}{\theta}\right)^{-\alpha}\right\},$$

where α is a shape parameter and θ is a scale parameter. In engineering applications shape parameter is usually greater than 2.

In a simple SSALT, all n products are initially placed on the test at a lower stress level S_1, and run until time τ when the stress is changed to S_2. The test is continued until all the products run to failure or until a predetermined censoring time T, whichever occurs first. S_0 is stress level at a typical operating condition. Such a test is called a simple step-stress test because it uses only two stress levels. Total n_i failures are observed at time t_{ij}, $j = 1, 2, ..., n_i$, while testing at stress level S_i, $i = 1, 2$, and $n_c = n - n_1 - n_2$ products remain unfailed and censored at time T.

2.1. Basic assumptions

The basic assumptions are:

1. Two stress levels S_1 and S_2 ($S_1 < S_2$) are used in the test.

2. For any level of stress, the life distribution of the test product follows a Fréchet distribution with the CDF (2.1).

3. The scale parameter θ_i at stress level i, $i = 0, 1, 2$ is a log-linear function of stress, *i.e.*,

$$\log(\theta_i) = \beta_0 + \beta_1 S_i$$

for $i = 0, 1, 2$, where β_0 and β_1 are unknown parameters depending on the nature of the product, and the method of test.

4. A cumulative exposure model holds, *i.e.*, the remaining life of a test product depends only on the cumulative exposure it has seen [10].

5. The lifetimes of the test products are identically distributed random variables.

From these assumptions, the CDF of a test product under simple step-stress test is

$$G(t) = \begin{cases} \exp\left\{-\left(\dfrac{t}{\theta_1}\right)^{-\alpha}\right\}, & 0 \le t < \tau\,, \\[2ex] \exp\left\{-\left(\dfrac{\tau}{\theta_1} + \dfrac{t-\tau}{\theta_2}\right)^{-\alpha}\right\}, & \tau \le t < \infty\,. \end{cases} \tag{2.2}$$

The corresponding PDF is

$$g(t) = \begin{cases} \dfrac{\alpha}{\theta_1}\left(\dfrac{t}{\theta_1}\right)^{-\alpha-1} \exp\left\{-\left(\dfrac{t}{\theta_1}\right)^{-\alpha}\right\}, & 0 \le t < \tau\,, \\[2ex] \dfrac{\alpha}{\theta_2}\left(\dfrac{\tau}{\theta_1} + \dfrac{t-\tau}{\theta_2}\right)^{-\alpha-1} \exp\left\{-\left(\dfrac{\tau}{\theta_1} + \dfrac{t-\tau}{\theta_2}\right)^{-\alpha}\right\}, & \tau \le t < \infty\,. \end{cases}$$

3. MAXIMUM LIKELIHOOD ESTIMATORS

The likelihood function under type I censoring can be written as

$$L(\theta_1, \theta_2, \alpha; t) = \prod_{j=1}^{n_1} g(t_{1j}) \prod_{j=1}^{n_2} g(t_{2j}) \left[1 - G(T)\right]^{n_c}.$$

Therefore,

$$\begin{aligned} L(\theta_1, \theta_2, \alpha; t) = \alpha^{n_1+n_2} &\left(\frac{1}{\theta_1}\right)^{n_1} \left(\frac{1}{\theta_2}\right)^{n_2} \prod_{j=1}^{n_1} \left(\frac{t_{1j}}{\theta_1}\right)^{-\alpha-1} \exp\left\{-\sum_{j=1}^{n_1}\left(\frac{t_{1j}}{\theta_1}\right)^{-\alpha}\right\} \\ &\cdot \prod_{j=1}^{n_2}\left(\frac{\tau}{\theta_1} + \frac{t_{2j}-\tau}{\theta_2}\right)^{-\alpha-1} \exp\left\{-\sum_{j=1}^{n_2}\left(\frac{\tau}{\theta_1} + \frac{t_{2j}-\tau}{\theta_2}\right)^{-\alpha}\right\} \\ &\cdot \left(1 - \exp\left\{-\left(\frac{\tau}{\theta_1} + \frac{T-\tau}{\theta_2}\right)^{-\alpha}\right\}\right)^{n_c}. \end{aligned}$$

It is usually easier to maximize the logarithm of the likelihood function rather than the likelihood function itself. The logarithm of the likelihood function is

$$\begin{aligned} \ell &= \log L(\theta_1, \theta_2, \alpha; t) \\ &= (n_1+n_2)\log\alpha - n_1\log\theta_1 - n_2\log\theta_2 \\ &\quad - (\alpha+1)\sum_{j=1}^{n_1}\log\left(\frac{t_{1j}}{\theta_1}\right) - \sum_{j=1}^{n_1}\left(\frac{t_{1j}}{\theta_1}\right)^{-\alpha} \\ &\quad - (\alpha+1)\sum_{j=1}^{n_2}\log\left(\frac{\tau}{\theta_1} + \frac{t_{2j}-\tau}{\theta_2}\right) - \sum_{j=1}^{n_2}\left(\frac{\tau}{\theta_1} + \frac{t_{2j}-\tau}{\theta_2}\right)^{-\alpha} \\ &\quad + n_c\log\left(1 - \exp\left\{-\left(\frac{\tau}{\theta_1} + \frac{T-\tau}{\theta_2}\right)^{-\alpha}\right\}\right). \end{aligned} \tag{3.1}$$

If at least one failure occurred before τ and T, MLEs of θ_1 and θ_2 do exist. In this case, MLEs of θ_1, θ_2 and α and hence the MLEs of β_0 and β_1 by the invariance property, they can be obtained through setting to zero the first partial derivatives of the log likelihood function with respect to θ_1, θ_2 and α. The system of equations is:

$$\begin{aligned}\frac{\partial \ell}{\partial \theta_1} &= \alpha\frac{n_1}{\theta_1} - \sum_{j=1}^{n_1}\frac{\alpha}{\theta_1}A_j^{-\alpha} + (\alpha+1)\sum_{j=1}^{n_2}\frac{\tau}{\theta_1^2}B_j^{-1} \\ &\quad -\frac{\alpha\tau}{\theta_1^2}\sum_{j=1}^{n_2}B_j^{-\alpha-1} - \frac{\alpha\, n_c \tau\,\theta_2}{\theta_1 E}C^{-\alpha}D^{-1} = 0\,,\end{aligned} \tag{3.2}$$

$$\begin{aligned}\frac{\partial \ell}{\partial \theta_2} &= -\frac{n_2}{\theta_2} + (\alpha+1)\sum_{j=1}^{n_2}\frac{t_{2j}-\tau}{\theta_2^2}B_j^{-1} - \alpha\sum_{j=1}^{n_2}\frac{t_{2j}-\tau}{\theta_2^2}B_j^{-\alpha-1} \\ &\quad -\frac{\alpha\, n_c (T-\tau)\,\theta_1}{\theta_2 D E}C^{-\alpha} = 0\,,\end{aligned} \tag{3.3}$$

$$\begin{aligned}\frac{\partial \ell}{\partial \alpha} &= \frac{n_1+n_2}{\alpha} - \sum_{j=1}^{n_1}\log A_j + \sum_{j=1}^{n_1}A_j^{-\alpha}\log A_j - \sum_{j=1}^{n_2}\log B_j \\ &\quad +\sum_{j=1}^{n_2}B_j^{-\alpha}\log B_j + n_c C^{-\alpha}D^{-1}\log C = 0\,,\end{aligned} \tag{3.4}$$

where $A_j = \frac{t_{1j}}{\theta_1}$, $j = 1, 2, ..., n_1$, $B_j = \frac{\tau}{\theta_1} + \frac{t_{2j}-\tau}{\theta_2}$, $j = 1, 2, ..., n_2$, $C = \frac{\tau}{\theta_1} + \frac{T-\tau}{\theta_2}$, $D = 1 - \exp\{C^{-\alpha}\}$ and $E = \theta_1(T-\tau) + \theta_2\tau$.

Given that, it is difficult to obtain a closed form solution to the nonlinear equations (3.2), (3.3) and (3.4), a numerical method is used to solve these equations. By solving these equations, the MLEs $(\theta_1, \theta_2, \alpha)$ and hence MLEs (β_0, β_1) can be obtained.

We have used from optimization tool in Matlab software for finding a maximum of a function of several variables.

The Fisher information essentially describes the amount of information data provide about an unknown parameter. It has applications in finding the variance of an estimator, as well as in the asymptotic behavior of maximum likelihood estimates. The inverse of the Fisher information matrix is an estimator of the asymptotic covariance matrix.

The Fisher information matrix $F(\theta_1, \theta_2, \alpha)$ is obtained through taking expectation on the negative of the second partial derivatives of $\ell(\theta_1, \theta_2, \alpha)$ with respect to θ_1, θ_2 and α.

$$F = n\begin{bmatrix} A_{11} & A_{12} & A_{13} \\ A_{12} & A_{22} & A_{23} \\ A_{13} & A_{23} & A_{33}\end{bmatrix}.$$

The calculation detail is presented in Appendix I and II. Therefore, the elements of F are given as

$$\begin{aligned}
A_{11} = \mathrm{E}\Big[-\frac{1}{n}\cdot\frac{\partial^2\ell}{\partial\theta_1^2}\Big] &= \frac{\alpha}{\theta_1^2}\,e_1 + \frac{\alpha(\alpha-1)}{\theta_1^2}\,I_1 + (\alpha+1)\,I_2 + I_3 \\
&- \alpha\tau\,(T-\tau)\,\frac{\theta_2\,C^{-\alpha}}{\theta_1 D E^2}\,e_c + \alpha^2\tau^2\,\frac{\theta_2\,C^{-1-2\alpha}(1-D)}{\theta_1^3 E D^2}\,e_c \\
&+ \alpha^2\tau^2\,\frac{\theta_2\,C^{-1-\alpha}}{\theta_1^3 E D}\,e_c - \alpha\tau\,\frac{\theta_2\,C^{-\alpha}}{\theta_1^2 E D}\,e_c\,,
\end{aligned}$$

$$\begin{aligned}
A_{22} = \mathrm{E}\Big[-\frac{1}{n}\cdot\frac{\partial^2\ell}{\partial\theta_2^2}\Big] &= -\frac{e_2}{\theta_2^2} + (\alpha+1)\,I_4 + I_5 + \frac{\alpha^2(T-\tau)^2\,\theta_1 C^{1-2\alpha}(1-D)}{\theta_2^3\,E D^2}\,e_c \\
&- \frac{\alpha\tau\,(T-\tau)\,\theta_1 C^{-\alpha}}{\theta_2\,E^2 D}\,e_c + \frac{\alpha^2(T-\tau)^2\,\theta_1 C^{-1-\alpha}}{\theta_2^3 E D}\,e_c \\
&- \frac{\alpha\,\theta_1(T-\tau)\,C^{-\alpha}}{\theta_2^2\,E D}\,e_c\,,
\end{aligned}$$

$$\begin{aligned}
A_{33} = \mathrm{E}\Big[-\frac{1}{n}\cdot\frac{\partial^2\ell}{\partial\alpha^2}\Big] &= \frac{1}{\alpha^2}\,e_1 + e_2 + I_6 + \frac{C^{-2\alpha}\,(\log C)^2\,(1-D)}{D^2}\,e_c \\
&+ \frac{C^{-\alpha}\,(\log C)^2}{D^2}\,e_c\,,
\end{aligned}$$

$$\begin{aligned}
A_{12} = \mathrm{E}\Big[-\frac{1}{n}\cdot\frac{\partial^2\ell}{\partial\theta_1\partial\theta_2}\Big] &= -\frac{\tau(\alpha+1)}{\theta_1^2}\,I_7 + \frac{\alpha(\alpha+1)\tau}{\theta_1^2}\,I_8 \\
&+ \alpha^2\tau\,(T-\tau)\,\frac{C^{-1-2\alpha}(1-D)}{\theta_1\theta_2\,E D^2}\,e_c - \alpha\tau^2\,\frac{\theta_2\,C^{-\alpha}}{\theta_1 E^2 D}\,e_c \\
&+ \alpha^2\tau\,(T-\tau)\,\frac{C^{-\alpha-1}}{\theta_1\theta_2\,E D}\,e_c + \alpha\tau\,\frac{C^{-\alpha}}{\theta_1 E D}\,e_c\,,
\end{aligned}$$

$$\begin{aligned}
A_{13} = \mathrm{E}\Big[-\frac{1}{n}\cdot\frac{\partial^2\ell}{\partial\theta_1\partial\alpha}\Big] &= -\frac{e_1}{\theta_1} + \frac{1}{\theta_1}\,I_9 + \frac{\tau}{\theta_1^2}\,I_{10} + \frac{\tau}{\theta_1^2}\,I_{11} + \tau\,\theta_2\,\frac{C^{-\alpha}}{\theta_1 E D}\,e_c \\
&- \alpha\tau\theta_2\,\frac{C^{-\alpha}\log C}{\theta_1 E D}\,e_c - \alpha\tau\theta_2\,\frac{C^{-2\alpha}(1-D)\log C}{\theta_1 E D^2}\,e_c\,,
\end{aligned}$$

$$\begin{aligned}
A_{23} = \mathrm{E}\Big[-\frac{1}{n}\cdot\frac{\partial^2\ell}{\partial\theta_2\partial\alpha}\Big] &= I_{12} + I_{13} - \alpha\,(T-\tau)\,\theta_1\,\frac{C^{-2\alpha}(1-D)\log C}{\theta_2\,E D^2}\,e_c \\
&+ \theta_1(T-\tau)\,\frac{C^{-\alpha}}{\theta_2\,E D}\,e_c - \alpha\,(T-\tau)\,\theta_1\,\frac{C^{\alpha}\log C}{\theta_2\,E D}\,e_c\,.
\end{aligned}$$

where the detailed calculation for I_1 to I_{13}, and e_1, e_2 and e_c in the formulas above are in Appendix I and Appendix II, respectively.

The asymptotic variance of the desired estimates is then obtained using the above Fisher information matrix, which leads to the optimization criteria.

4. OPTIMUM TEST DESIGN

As mentioned earlier, for the purpose of optimization, two criteria are considered. The first criterion (Criterion I) is minimizing the asymptotic variance (AV) of the MLE of the logarithm of the percentile life under usual operating conditions, which is used when the percentile life is the desired estimate. Furthermore, we can minimize the AV of reliability estimate at time ξ under usual operating conditions. We call this criterion as the second criterion (Criterion II) and is used when we want to predict reliability.

We will show the optimal hold times achieved by criterion I and II, with the symbols of τ^* and τ^+, respectively.

4.1. Criterion I

As mentioned above, in this criterion, we try to minimize the AV of the MLE of the logarithm of percentile life under the usual operating conditions. This is the most commonly used criterion.

The reliability function at time t under the usual operating condition, S_0, is:

$$R_0(t) \;=\; 1 - G_0(t) \;=\; 1 - \exp\left\{-\left(\frac{t}{\theta_0}\right)^{-\alpha}\right\}.$$

For a specified reliability R, the $100(1-R)$-th percentile life under the usual operating condition, S_0, is:

$$t_R \;=\; \theta_0\Big(-\log(1-R)\Big)^{-1/\alpha}.$$

From assumption 3 and the definition, $x = \frac{S_1 - S_0}{S_2 - S_0}$, we obtain $S_0 = \frac{S_1 - xS_2}{1-x}$, thus,

$$\log\theta_0 \;=\; \frac{\log\theta_1 - x\log\theta_2}{1-x}\,. \tag{4.1}$$

Therefore, the MLE of the log of the $100(1-R)$-th percentile life of the Fréchet distribution with a specified reliability R under the usual operating condition, S_0, is:

$$\begin{aligned}\log\big(\widehat{t}_R\big) \;&=\; \log\widehat{\theta}_0 - \frac{1}{\widehat{\alpha}}\log\Big(-\log(1-R)\Big)\\ &=\; \frac{\log\widehat{\theta}_1 - x\log\widehat{\theta}_2}{1-x} - \frac{\log\Big(-\log(1-R)\Big)}{\widehat{\alpha}}\,.\end{aligned}$$

The optimality criterion used for the SSALT design is to minimize the AV of the MLE of the log of the $100(1-R)$-th percentile life of the Fréchet distribution at S_0 with a specified reliability R. When $R=0.5$, $\log(\widehat{t}_R)$ is the logarithm of the median life at usual operating conditions with stress level S_0. To obtain the $AV[\log(\widehat{t}_R)]$, we use the delta method which described in Appendix III.

The optimal hold time τ_0^* at which $AV[\log(\widehat{t}_R)]$ reaches its minimum value leads to the optimal plan:

$$AV\Big[\log\big(\widehat{t}_R\big)\Big] \;=\; AV\left[\frac{\log\widehat{\theta}_1 - x\log\widehat{\theta}_2}{1-x} - \frac{\log\Big(-\log\,(1-R)\Big)}{\widehat{\alpha}}\right] \;=\; H_1\,\widehat{F}^{-1}H_1'\;,$$

where $\widehat{F}$ is estimated the Fisher information matrix and H_1 is the row vector of the first derivative of $\log(\widehat{t}_R)$ with respect to $\widehat{\theta}_1$, $\widehat{\theta}_2$ and $\widehat{\alpha}$; and in practice, the values of $(\widehat{\theta}_1,\widehat{\theta}_2,\widehat{\alpha})$ are obtained from a previous experience based on a similar data, or based on a preliminary test result.

$$H_1 \;=\; \left[\frac{1}{\widehat{\theta}_1(1-x)},\ \frac{x}{\widehat{\theta}_2\,(x-1)},\ \frac{\log\Big(-\log\,(1-R)\Big)}{\widehat{\alpha}^2}\right].$$

4.2. Criterion II

Reliability prediction is an important factor in a product design and during the developmental testing process. In order to accurately estimate the product reliability, the test design criterion is defined to minimize the AV of the reliability estimate at a time ξ under the normal operating conditions.

The MLE of reliability at ξ from the Fréchet distribution at the usual operating stress level, S_0, is:

$$\widehat{R}_{S_0}(\xi) \;=\; 1-\exp\left\{-\left(\frac{\xi}{\widehat{\theta}_0}\right)^{-\widehat{\alpha}}\right\} \;=\; 1-\exp\left\{-\exp\left\{-\,\widehat{\alpha}\log\xi+\widehat{\alpha}\log\widehat{\theta}_0\right\}\right\},$$

where, by using (4.1), we have

$$\widehat{R}_{S_0}(\xi) \;=\; 1-\exp\left\{-\exp\left\{-\,\widehat{\alpha}\log\xi+\widehat{\alpha}\,\frac{\log\widehat{\theta}_1 - x\log\widehat{\theta}_2}{1-x}\right\}\right\}.$$

The AV of the reliability estimate at time ξ under normal operating conditions, by using the delta method, can be obtained as:

$$\begin{aligned}(4.2)\qquad AV\Big[\widehat{R}_{S_0}(\xi)\Big] \;&=\; AV\left[1-\exp\left\{-\exp\left\{-\,\widehat{\alpha}\log\xi+\widehat{\alpha}\log\widehat{\theta}_0\right\}\right\}\right]\\ &=\; H_2\,\widehat{F}^{-1}H_2'\;,\end{aligned}$$

where H_2 is the row vector of the first derivative of $\widehat{R}_{S_0}(\xi)$ with respect to $\widehat{\theta}_1$, $\widehat{\theta}_2$ and $\widehat{\alpha}$, *i.e.*, $H_2 = \Big[H_{11}, H_{12}, H_{13}\Big]$, where its components are given below. In practice, Based on experience, some historical data or a preliminary test can be used to get the values of $(\widehat{\theta}_1, \widehat{\theta}_2, \widehat{\alpha})$.

$$
\begin{aligned}
H_{11} &= \frac{\widehat{\alpha}\,\xi^{-\widehat{\alpha}}}{\widehat{\theta}_1(1-x)} \exp\left\{-\,\xi^{-\widehat{\alpha}}\, e^{\widehat{\alpha}\frac{\log\widehat{\theta}_1 - x\log\widehat{\theta}_2}{1-x}} + \widehat{\alpha}\,\frac{\log\widehat{\theta}_1 - x\log\widehat{\theta}_2}{1-x}\right\},\\
H_{12} &= \frac{x\,\widehat{\alpha}\,\xi^{-\widehat{\alpha}}}{\widehat{\theta}_2\,(x-1)} \exp\left\{-\,\xi^{-\widehat{\alpha}}\, e^{\widehat{\alpha}\frac{\log\widehat{\theta}_1 - x\log\widehat{\theta}_2}{1-x}} + \widehat{\alpha}\,\frac{\log\widehat{\theta}_1 - x\log\widehat{\theta}_2}{1-x}\right\},\\
H_{13} &= \frac{1}{x-1}\left(\exp\left\{-\,\xi^{-\widehat{\alpha}}\, e^{\widehat{\alpha}\frac{\log\widehat{\theta}_1 - x\log\widehat{\theta}_2}{1-x}} + \widehat{\alpha}\,\frac{\log\widehat{\theta}_1 - x\log\widehat{\theta}_2}{1-x}\right\}\right.\\
&\qquad \left.\cdot\,\xi^{-\widehat{\alpha}}\left(\log\xi - x\log\xi - \log\widehat{\theta}_1 + x\log\widehat{\theta}_2\right)\right).
\end{aligned}
$$

The value τ_0^+ that minimizes $AV\Big[\widehat{R}_{S_0}(\xi)\Big]$, given by equation (4.2), leads to the optimal SSALT plan.

4.3. Simulation study

The main objective of this simulation study is numerical investigation for illustrating the theoretical results of both estimation and optimal design problems given in this paper. Considering type I censoring, data were generated from Fréchet distribution under SSALT for different combinations of the true parameter values of θ_1, θ_2 and α. The true parameters values used here are $(1.5, 1, 1)$ and $(2.5, 2, 1.5)$. In addition, $\tau = 2.5$ and $T = 5$ have been considered. The samples sizes considered are $n = 100, 200, 300, 400, 500, 1000$ each with ten thousand replications. A numerical method is used for the MLEs of θ_1, θ_2 and α. The nonlinear likelihood equations, (3.2), (3.3) and (3.4), were solved iteratively. The MLEs, their mean square errors (MSEs) and their relative errors (REs) are reported in Table 1 for different sample sizes and different true values of the parameters. The results provide insight into the sampling behavior of the estimators. They indicate that the MLEs approximate the true values of the parameters as the sample size n increases. Similarly, the MSEs and REs decrease with increasing the sample size.

To illustrate the procedure of the optimum test design, we proposed a standardized model. A standardized censoring time $T_0 = 1$ is assumed, and the standardized scale parameter $\eta_i = \frac{\theta_i}{T}$ is defined. The standardized hold time τ_0 is also defined as the ratio of the hold time to the censoring time $\tau_0 = \frac{\tau}{T}$. Thus the value of τ_0 that minimizes AV is the optimal standardized hold time, and the optimal hold time is derived from $\tau^* = \tau_0^* \cdot T$ and $\tau^+ = \tau_0^+ \cdot T$, with respect to criterion I and II. Using the standardized model, we eliminate the input value of censoring time and embed it in the standardized scale parameters.

Table 1: The MLEs of the parameters, and the associated MSE and RE for different sample sizes.

n	Parameter	$(\theta_1=1.5,\ \theta_2=1,\ \alpha=1)$			$(\theta_1=2.5,\ \theta_2=2,\ \alpha=1.5)$		
		Estimate	MSE	RE	Estimate	MSE	RE
$n=100$	θ_1	1.4404	0.0108	0.0397	2.5228	0.0500	0.0091
	θ_2	1.0381	0.0771	0.0381	2.1101	0.2825	0.0091
	α	1.0307	0.0103	0.0307	1.5362	0.0367	0.0241
$n=200$	θ_1	1.4563	0.0058	0.0291	2.5086	0.0238	0.0034
	θ_2	1.0273	0.0461	0.0273	2.0541	0.1372	0.0271
	α	1.0205	0.0050	0.0205	1.5180	0.0186	0.0120
$n=300$	θ_1	1.4640	0.0040	0.0240	2.5061	0.0156	0.0024
	θ_2	1.0196	0.0317	0.0196	2.0374	0.0877	0.0187
	α	1.0160	0.0033	0.0160	1.5128	0.0121	0.0085
$n=400$	θ_1	1.4682	0.0031	0.0212	2.5044	0.0118	0.0018
	θ_2	1.0142	0.0237	0.0142	2.0279	0.0650	0.0140
	α	1.0133	0.0025	0.0133	1.5094	0.0090	0.0063
$n=500$	θ_1	1.4703	0.0026	0.0198	2.5036	0.0095	0.0014
	θ_2	1.0132	0.0194	0.0132	2.0200	0.0501	0.0100
	α	1.0177	0.0020	0.0117	1.5073	0.0071	0.0049
$n=1000$	θ_1	1.4801	0.0012	0.0132	2.5021	0.0046	8.4513×10^{-4}
	θ_2	1.0055	0.0094	0.0055	2.0106	0.0249	0.0053
	α	1.0073	9.5334×10^{-4}	0.0073	1.5034	0.0036	0.0023

Now, the numerical examples are given for calculating the optimal standardized hold times of the simple SSALT under both criteria.

In the first example, we suppose that a simple SSALT to estimate the percentile life of the Fréchet distribution under the usual operating condition with a specified reliability R. For the given values of $\theta_1 = 900$, $\theta_2 = 400$, $\alpha = 2$, $T = 1000$, $x = 0.5$ and assuming $R = 0.5$, we determine the optimal hold time τ^*. Based on the above transformation, the standardized parameters are obtained as $\eta_1 = 0.9$ and $\eta_2 = 0.4$. Using the criterion I, the optimal standardized hold time is obtained $\tau_0^* = 0.8165$. So, the optimum stress change time is obtained $\tau^* = 816.5$.

Sensitivity analysis is performed to examine the effect of the changes in the pre-estimated parameters $(\theta_1, \theta_2, \alpha)$ on the optimal hold time τ. Its objective is to identify the sensitive parameters, which need to be estimated with special care to minimize the risk of obtaining an erroneous optimal solution. According to the definition of x and R; and since they take different values, we also examine the impact of changes in their values.

Table 2 presents the standardized optimal hold time for the specified values of $n = 30$, $R = 0.5$, $x = 0.5$, $\alpha = 2$, $\eta_1 = 0.3, 0.5, ..., 1.7$ and $\eta_2 = 0.1, 0.3, ..., 1.5$. From this table, we can see that as η_1 increases, the optimal standardized stress change time slightly increases. And also, as η_2 increases, then slightly decreases.

Table 2: Optimal standardized hold time τ_0^* and AV versus changes in η_1 and η_2 with criterion I ($\alpha = 2$, $x = 0.5$, $R = 0.5$).

η_1	η_2							
	0.1	0.3	0.5	0.7	0.9	1.1	1.3	1.5
0.3	$\tau_0^* = 0.3625$ AV = 8.9925							
0.5	$\tau_0^* = 0.6085$ AV = 9.0273	$\tau_0^* = 0.5985$ AV = 9.0828						
0.7	$\tau_0^* = 0.8385$ AV = 9.0832	$\tau_0^* = 0.7635$ AV = 9.4182	$\tau_0^* = 0.7165$ AV = 9.8620					
0.9	$\tau_0^* = 0.9175$ AV = 9.6894	$\tau_0^* = 0.8435$ AV = 10.3295	$\tau_0^* = 0.7925$ AV = 10.8826	$\tau_0^* = 0.7475$ AV = 11.3462				
1.1	$\tau_0^* = 0.9425$ AV = 10.9082	$\tau_0^* = 0.8725$ AV = 11.4226	$\tau_0^* = 0.8215$ AV = 11.8557	$\tau_0^* = 0.7865$ AV = 12.2794	$\tau_0^* = 0.7625$ AV = 12.7426			
1.3	$\tau_0^* = 0.9495$ AV = 11.9406	$\tau_0^* = 0.8965$ AV = 12.4060	$\tau_0^* = 0.8605$ AV = 13.0544	$\tau_0^* = 0.8355$ AV = 13.8146	$\tau_0^* = 0.8165$ AV = 14.6650	$\tau_0^* = 0.8025$ AV = 15.5903		
1.5	$\tau_0^* = 0.9495$ AV = 13.4693	$\tau_0^* = 0.9195$ AV = 14.3669	$\tau_0^* = 0.8915$ AV = 15.6970	$\tau_0^* = 0.8705$ AV = 17.1575	$\tau_0^* = 0.8545$ AV = 18.7231	$\tau_0^* = 0.8425$ AV = 20.3801	$\tau_0^* = 0.8325$ AV = 22.1190	
1.7	$\tau_0^* = 0.9495$ AV = 17.3690	$\tau_0^* = 0.9355$ AV = 18.7619	$\tau_0^* = 0.9115$ AV = 21.3211	$\tau_0^* = 0.8935$ AV = 24.0302	$\tau_0^* = 0.8805$ AV = 26.8841	$\tau_0^* = 0.8695$ AV = 29.8769	$\tau_0^* = 0.8605$ AV = 33.0031	$\tau_0^* = 0.8535$ AV = 36.2555

Figure 1 shows the sensitivity of the initially estimated parameters with respect to the criterion I. We can see:

1. The optimal value of τ^*, slightly increases as η_1 and α increase for smaller values of η_1 and α, and converges for larger values of η_1 and α;

2. The optimal value of τ^*, slightly decreases as η_2, R and x increase, and it is not too sensitive to parameters η_2 and x.

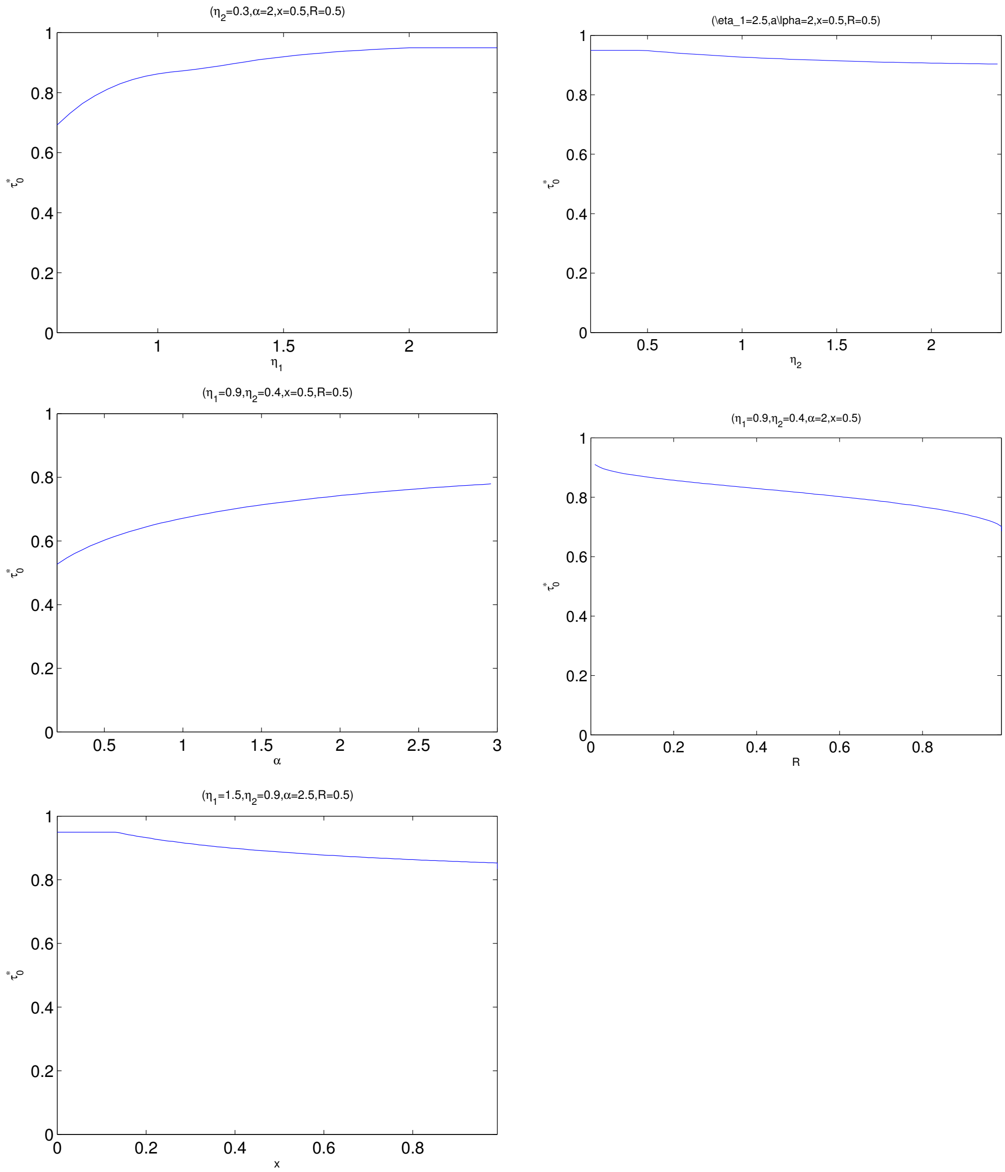

Figure 1: Optimal standardized hold time versus changes in initial parameters under criterion I.

Table 3: Optimal standardized hold time τ_0^+ and AV versus changes in η_1 and η_2 with criterion II ($\alpha = 2$, $x = 0.5$, $\xi = 10$).

η_1	η_2							
	0.1	0.3	0.5	0.7	0.9	1.1	1.3	1.5
0.3	$\tau_0^+ = 0.5155$ AV $= 0.0068$							
0.5	$\tau_0^+ = 0.7595$ AV $= 0.2379$	$\tau_0^+ = 0.7595$ AV $= 0.0054$						
0.7	$\tau_0^+ = 0.8815$ AV $= 1.7952$	$\tau_0^+ = 0.8585$ AV $= 0.0649$	$\tau_0^+ = 0.8415$ AV $= 0.0116$					
0.9	$\tau_0^+ = 0.9195$ AV $= 4.5875$	$\tau_0^+ = 0.8995$ AV $= 0.4237$	$\tau_0^+ = 0.8865$ AV $= 0.0829$	$\tau_0^+ = 0.8765$ AV $= 0.0274$				
1.1	$\tau_0^+ = 0.9275$ AV $= 3.5379$	$\tau_0^+ = 0.9065$ AV $= 1.5776$	$\tau_0^+ = 0.8875$ AV $= 0.3703$	$\tau_0^+ = 0.8695$ AV $= 0.1287$	$\tau_0^+ = 0.8535$ AV $= 0.0568$			
1.3	$\tau_0^+ = 0.9335$ AV $= 0.6480$	$\tau_0^+ = 0.9075$ AV $= 3.6960$	$\tau_0^+ = 0.8845$ AV $= 1.0889$	$\tau_0^+ = 0.8685$ AV $= 0.4059$	$\tau_0^+ = 0.8555$ AV $= 0.1851$	$\tau_0^+ = 0.8455$ AV $= 0.0970$		
1.5	$\tau_0^+ = 0.9405$ AV $= 0.0226$	$\tau_0^+ = 0.9225$ AV $= 6.4165$	$\tau_0^+ = 0.9065$ AV $= 2.6665$	$\tau_0^+ = 0.8955$ AV $= 1.1107$	$\tau_0^+ = 0.8865$ AV $= 0.5355$	$\tau_0^+ = 0.8795$ AV $= 0.2908$	$\tau_0^+ = 0.8735$ AV $= 0.1727$	
1.7	$\tau_0^+ = 0.9445$ AV $= 9.4376\times10^{-5}$	$\tau_0^+ = 0.9315$ AV $= 9.0884$	$\tau_0^+ = 0.9205$ AV $= 6.1714$	$\tau_0^+ = 0.9115$ AV $= 2.9891$	$\tau_0^+ = 0.9045$ AV $= 1.5486$	$\tau_0^+ = 0.8995$ AV $= 0.8780$	$\tau_0^+ = 0.8945$ AV $= 0.5370$	$\tau_0^+ = 0.8905$ AV $= 0.3491$

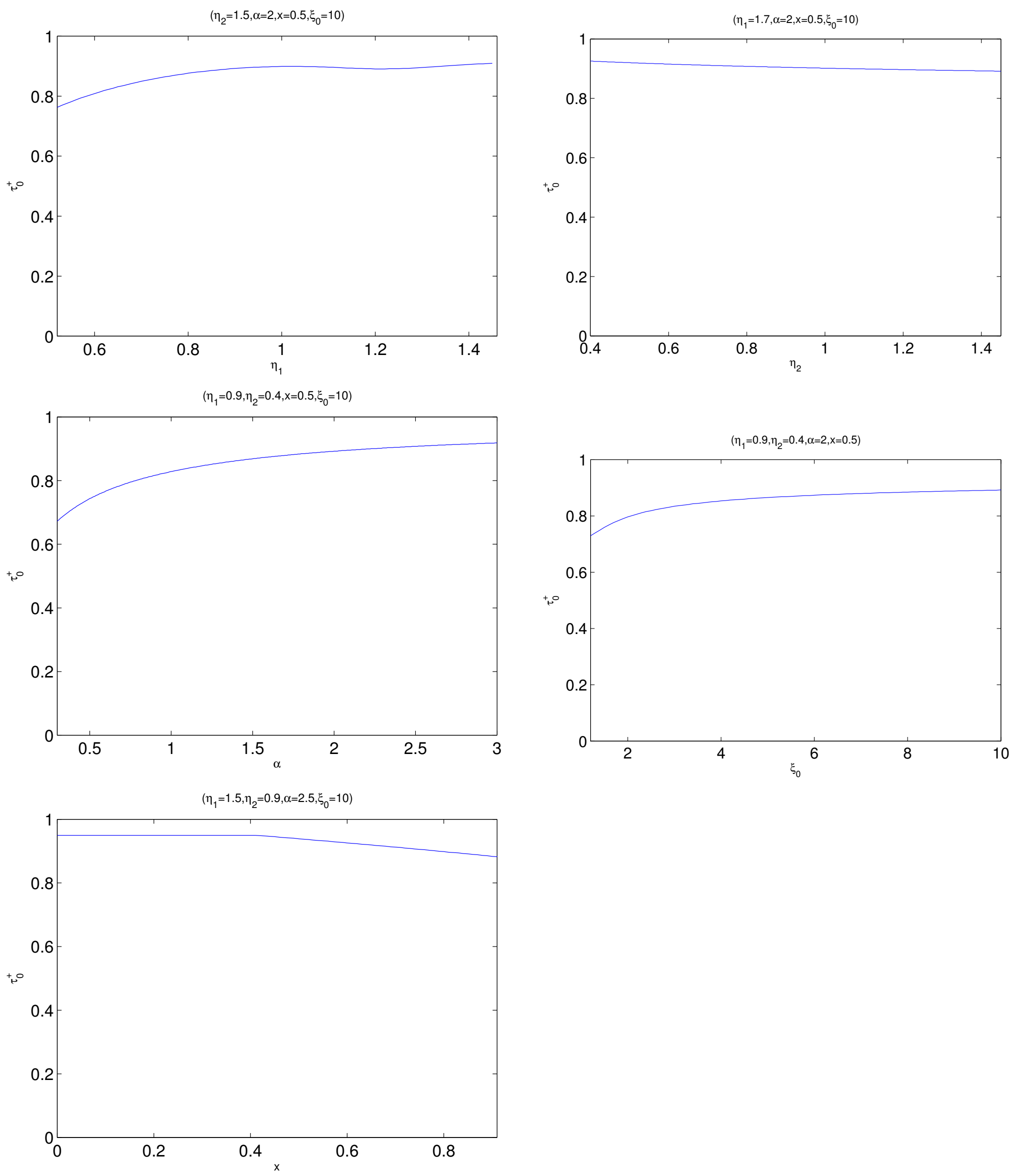

Figure 2: Optimal standardized hold time versus changes in initial parameters under criterion II.

In the second example, we suppose a simple SSALT is run to estimate the reliability at a specified time $\xi = 10000$. The objective is to design a test that achieves the best reliability estimates. To obtain the optimal hold time τ^+, the AV of the reliability estimate at time ξ is minimized. The initial parameters given $\theta_1 = 900,\ \theta_2 = 400,\ \alpha = 2,\ x = 0.5,\ T = 1000$ and $\xi = 10000$. Then the standardized parameters are obtained as $\eta_1 = 0.9,\ \eta_2 = 0.4$ and $\xi_0 = 10$. By criterion II, the optimum standardized hold time is obtained as $\tau_0^+ = 0.8925$ and the optimum stress change time is obtained as $\tau^+ = 8.925$.

Table 3 presents the standardized optimal hold time for the specified values of $n = 30,\ x = 0.5,\ \alpha = 2,\ \xi = 10,\ \eta_1 = 0.3, 0.5, ..., 1.7$ and $\eta_2 = 0.1, 0.3, ..., 1.5$. This table shows that, as η_1 increases, the optimal standardized stress change time slightly increases. And also, as η_2 increases, then slightly decreases.

Figure 2 shows the sensitivity of the initially estimated parameters with respect to criterion II. We can see:

1. The optimal value of τ^+, slightly increases as η_1, α and ξ increase for smaller values of them, and converges for larger values of them;

2. The optimal value of τ^+, slightly decreases as η_2 and x increase, and is not too sensitive to parameters η_2 and x.

5. CONCLUSION

In this paper, we proposed an optimal design of simple step stress accelerated life test with type I censored Fréchet data. Optimizing test plan will lead to an improved parameter estimation which would further lead to a higher quality of inference. The estimation was based on the maximum likelihood.

For the purpose of optimizing, two criteria were considered. These criteria were based on minimizing the AV of the life estimate and the reliability estimate. Furthermore, according to the simulation study, we have found that since the optimal hold times are not too sensitive to the model parameters, thus the proposed design is robust. The results show that the simple SSALT model can be reliably used which would remove the need for examining all the test products and would have economic benefits concerning time and money.

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APPENDIX I

The Fisher information matrix, can be obtained by taking the expected values of the negative second derivatives with respect to θ_1, θ_2 and α of the function (3.1). The results of these derivatives are given the following:

$$
\begin{aligned}
I_{11} = -\frac{\partial^2 \ell}{\partial \theta_1^2} = {} & \frac{\alpha n_1}{\theta_1^2} + \sum_{j=1}^{n_1} \left(\frac{\alpha(\alpha+1) A_j^{-\alpha}}{\theta_1^2} - \frac{2\alpha A_j^{-\alpha}}{\theta_1^2} \right) \\
& + (\alpha+1) \sum_{j=1}^{n_2} \left(-\frac{\tau^2}{\theta_1^4 B_j^2} - \frac{2\tau}{\theta_1^3 B_j} \right) \\
& + \sum_{j=1}^{n_2} \left(\frac{\alpha(\alpha+1)\tau^2 B_j^{-2-\alpha}}{\theta_1^4} - \frac{2\alpha\tau B_j^{-1-\alpha}}{\theta_1^3} \right) \\
& - \alpha\tau(T-\tau)\frac{\theta_2 C^{-\alpha}}{\theta_1 D E^2} n_c + \alpha^2\tau^2 \frac{\theta_2 C^{-1-2\alpha}(1-D)}{\theta_1^3 E D^2} n_c \\
& + \alpha^2\tau^2 \frac{\theta_2 C^{-1-\alpha}}{\theta_1^3 E D} n_c - \alpha\tau n_c \frac{\theta_2 C^{-\alpha}}{\theta_1^2 E D} \,,
\end{aligned}
$$

$$
\begin{aligned}
I_{22} = -\frac{\partial^2 \ell}{\partial \theta_2^2} = {} & -\frac{n_2}{\theta_2^2} + \alpha \sum_{j=1}^{n_2} \left((\alpha+1) F_j^2 B_j^{-2-\alpha} - \frac{2}{\theta_2} B_j^{-\alpha-1} F_j \right) \\
& + (\alpha+1) \sum_{j=1}^{n_2} \left(-\frac{F_j^2}{\theta_2^2 B_j^2} + \frac{2F_j}{\theta_2 B_j} \right) \\
& + \frac{\alpha^2 (T-\tau)^2 \theta_1 C^{1-2\alpha}(1-D)}{\theta_2^3 E D^2} n_c - \frac{\alpha\tau(T-\tau)\theta_1 C^{-\alpha}}{\theta_2 E^2 D} n_c \\
& + \frac{\alpha^2 (T-\tau)^2 \theta_1 C^{-1-\alpha}}{\theta_2^3 E D} n_c - \frac{\alpha\theta_1 (T-\tau) C^{-\alpha}}{\theta_2^2 E D} n_c \,,
\end{aligned}
$$

$$
\begin{aligned}
I_{33} = -\frac{\partial^2 \ell}{\partial \alpha^2} = {} & \frac{n_1 + n_2}{\alpha^2} + \sum_{j=1}^{n_2} B_j^{-\alpha} \left(\log B_j \right)^2 \\
& + \frac{C^{-2\alpha} (\log C)^2 (1-D)}{D^2} n_c + \frac{C^{-\alpha} (\log C)^2}{D^2} n_c \,,
\end{aligned}
$$

$$
\begin{aligned}
I_{12} = -\frac{\partial^2 \ell}{\partial \theta_1 \partial \theta_2} = {} & -\frac{(\alpha+1)\tau}{\theta_1^2} \sum_{j=1}^{n_2} \frac{F_j}{\theta_1^2 B_j^2} + \frac{\alpha(\alpha+1)\tau}{\theta_1^2} \sum_{j=1}^{n_2} F_j B_j^{-2-\alpha} \\
& + \alpha\tau \frac{C^{-\alpha}}{\theta_1 E D} n_c + \alpha^2 \tau (T-\tau) \frac{C^{-1-2\alpha}(1-D)}{\theta_1 \theta_2 E D^2} n_c \\
& - \alpha\tau^2 \frac{\theta_2 C^{-\alpha}}{\theta_1 E^2 D} n_c + \alpha^2 \tau (T-\tau) \frac{C^{-\alpha-1}}{\theta_1 \theta_2 E D} n_c \,,
\end{aligned}
$$

$$\begin{aligned}
I_{13} = -\frac{\partial^2 \ell}{\partial\theta_1\partial\alpha} &= -\frac{n_1}{\theta_1} + \frac{1}{\theta_1}\sum_{j=1}^{n_1} A_j^{-\alpha} - \sum_{j=1}^{n_2} \frac{\tau}{\theta_1^2 B_j} \\
&+ \frac{\tau}{\theta_1^2}\sum_{j=1}^{n_2} B_j^{-1-\alpha}\Big(1-\alpha\log B_j\Big) + \tau\,\theta_2\,\frac{C^{-\alpha}}{\theta_1 ED}\,n_c \\
&- \alpha\tau\,\theta_2\,\frac{C^{-2\alpha}(1-D)\log C}{\theta_1 ED^2}\,n_c - \alpha\tau\,\theta_2\,\frac{C^{-\alpha}\log C}{\theta_1 ED}\,n_c\ ,
\end{aligned}$$

$$\begin{aligned}
I_{23} = -\frac{\partial^2 \ell}{\partial\theta_2\partial\alpha} &= \sum_{j=1}^{n_2}\frac{F_j}{B_j} + \sum_{j=1}^{n_2} F_j\,B_j^{-1-\alpha}\Big(1-\alpha\log B_j\Big) \\
&- \alpha\,(T-\tau)\,\theta_1\,\frac{C^{-2\alpha}(1-D)\log C}{\theta_2\,ED^2}\,n_c \\
&+ \theta_1(T-\tau)\,\frac{C^{-\alpha}}{\theta_2\,ED}\,n_c - \alpha\,(T-\tau)\,\theta_1\,\frac{C^{\alpha}\log C}{\theta_2\,ED}\,n_c\ ,
\end{aligned}$$

where $F_j = \frac{t_{2j}-\tau}{\theta_2^2}$, $j = 1, 2, ..., n_2$.

The results of the above equations are then used to develop the Fisher information matrix. And also, to simplify the second partial and mixed partial derivatives, the following definitions are made:

$$I_1 = \mathrm{E}\left[\frac{1}{n}\sum_{j=1}^{n_1} A_j^{-\alpha}\right] = \int_0^{\tau} A_j^{-\alpha} g(t)\,d(t)\ ,$$

$$I_2 = \mathrm{E}\left[\frac{1}{n}\sum_{j=1}^{n_2}\left(-\frac{\tau^2}{\theta_1^4 B_j^2} - \frac{2\tau}{\theta_1^3 B_j}\right)\right] = \int_{\tau}^{T}\left(-\frac{\tau^2}{\theta_1^4 B_j^2} - \frac{2\tau}{\theta_1^3 B_j}\right) g(t)\,d(t)\ ,$$

$$\begin{aligned}
I_3 &= \mathrm{E}\left[\frac{1}{n}\sum_{j=1}^{n_2}\left(\frac{\alpha\,(\alpha+1)\,\tau^2\,B_j^{-2-\alpha}}{\theta_1^4} - \frac{2\,\alpha\,\tau\,B_j^{-1-\alpha}}{\theta_1^3}\right)\right] \\
&= \int_{\tau}^{T}\left(\frac{\alpha\,(\alpha+1)\,\tau^2\,B_j^{-2-\alpha}}{\theta_1^4} - \frac{2\,\alpha\,\tau\,B_j^{-1-\alpha}}{\theta_1^3}\right) g(t)\,d(t)\ ,
\end{aligned}$$

$$I_4 = \mathrm{E}\left[\frac{1}{n}\sum_{j=1}^{n_2}\left(-\frac{F_j^2}{\theta_2^2 B_j^2} + \frac{2\,F_j}{\theta_2\,B_j}\right)\right] = \int_{\tau}^{T}\left(-\frac{F_j^2}{\theta_2^2 B_j^2} + \frac{2\,F_j}{\theta_2\,B_j}\right) g(t)\,d(t)\ ,$$

$$\begin{aligned}
I_5 &= \mathrm{E}\left[\frac{1}{n}\sum_{j=1}^{n_2}\left((\alpha+1)\,F_j^2\,B_j^{-2-\alpha} - \frac{2}{\theta_2}\,B_j^{-\alpha-1}F_j\right)\right] \\
&= \int_{\tau}^{T}\left((\alpha+1)\,F_j^2\,B_j^{-2-\alpha} - \frac{2}{\theta_2}\,B_j^{-\alpha-1}F_j\right) g(t)\,d(t)\ ,
\end{aligned}$$

$$I_6 = \mathrm{E}\left[\frac{1}{n}\sum_{j=1}^{n_2} B_j^{-\alpha}\Big(\log B_j\Big)^2\right] = \int_{\tau}^{T} B_j^{-\alpha}\Big(\log B_j\Big)^2 g(t)\,d(t)\ ,$$

$$
\begin{aligned}
I_7 &= \mathrm{E}\left[\frac{1}{n}\sum_{j=1}^{n_2}\frac{F_j}{\theta_1^2 B_j^2}\right] = \int_{\tau}^{T}\frac{F_j}{\theta_1^2 B_j^2}\; g(t)\,d(t)\;, \\
I_8 &= \mathrm{E}\left[\frac{1}{n}\sum_{j=1}^{n_2} F_j\,B_j^{-2-\alpha}\right] = \int_{\tau}^{T} F_j\,B_j^{-2-\alpha}\,g(t)\,d(t)\;, \\
I_9 &= \mathrm{E}\left[\frac{1}{n}\sum_{j=1}^{n_1} A_j^{-\alpha}\right] = \int_{0}^{\tau} A_j^{-\alpha}\,g(t)\,d(t)\;, \\
I_{10} &= \mathrm{E}\left[\frac{1}{n}\sum_{j=1}^{n_2} B_j^{-1-\alpha}\Big(1-\alpha\log B_j\Big)\right] = \int_{\tau}^{T} B_j^{-1-\alpha}\Big(1-\alpha\log B_j\Big)\,g(t)\,d(t)\;, \\
I_{11} &= \mathrm{E}\left[\frac{1}{n}\sum_{j=1}^{n_2}\frac{1}{B_j}\right] = \int_{\tau}^{T}\frac{1}{B_j}\; g(t)\,d(t)\;, \\
I_{12} &= \mathrm{E}\left[\frac{1}{n}\sum_{j=1}^{n_2}\frac{F_j}{B_j}\right] = \int_{\tau}^{T}\frac{F_j}{B_j}\; g(t)\,d(t)\;, \\
I_{13} &= \mathrm{E}\left[\frac{1}{n}\sum_{j=1}^{n_2} F_j\,B_j^{-1-\alpha}\Big(1-\alpha\log B_j\Big)\right] \\
&= \int_{\tau}^{T}\sum_{j=1}^{n_2} F_j\,B_j^{-1-\alpha}\Big(1-\alpha\log B_j\Big)\,g(t)\,d(t)\;.
\end{aligned}
$$

APPENDIX II

Detailed calculations of $e_i = \mathrm{E}\left[\frac{n_i}{n}\right]$, $i = 1, 2$ is demonstrated through the following three steps:

At the first step, n new products are tested at stress levels S_1 until time τ, where the test units are assumed independent and identically distributed. The life of items follows the CDF of t in equation (2.2). The number of failures n_1 in time τ is a binomial random variable with parameters n and p_1. From the equation (2.3), we have:

$$\begin{aligned} p_1 &= G(\tau) = \exp\left\{-\left(\frac{\tau}{\theta_1}\right)^{-\alpha}\right\}, \\ e_1 &= \mathrm{E}\left[\frac{n_1}{n}\right] = p_1 = \exp\left\{-\left(\frac{\tau}{\theta_1}\right)^{-\alpha}\right\}. \end{aligned}$$

The second step starts with $n - n_1$ unfailed items, tested at stress levels S_2 until time T. The life of items follows the CDF of t given by the equation (2.2), where the number of failures n_2 follows a binomial distribution with parameters $n - n_1$ and p_2. Then, from the equation (2.2), we have:

$$\begin{aligned} p_2 &= P_r\Big(\textit{item fails in time } T \;\Big|\; \textit{it not fails in time } \tau \textit{ in first step}\Big) \\ &= 1 - P_r\Big(\textit{item not fails in item } T \;\Big|\; \textit{item not fails in time } \tau\Big) \\ &= \frac{\exp\left\{-\left(\frac{\tau}{\theta_1} + \frac{T-\tau}{\theta_2}\right)^{-\alpha}\right\} - \exp\left\{-\left(\frac{\tau}{\theta_1}\right)^{-\alpha}\right\}}{1 - \exp\left\{-\left(\frac{\tau}{\theta_1}\right)^{-\alpha}\right\}}, \\ e_2 &= \mathrm{E}\left[\frac{n_2}{n}\right] = \mathrm{E}\left[\frac{n_2}{n-n_1} \cdot \frac{n-n_1}{n}\right] = p_2 \cdot (1 - p_1). \end{aligned}$$

APPENDIX III

In statistics, the delta method is a result concerning the approximate probability distribution for a function of an asymptotically normal statistical estimator from knowledge of the limiting variance of that estimator.

A consistent estimator B converges in probability to its true value β, and often a central limit theorem can be applied to obtain asymptotic normality:

$$\sqrt{n}(B-\beta) \xrightarrow{D} N(0,\Sigma) ,$$

where n is the number of observations and $\sum$ is a (symmetric positive semi-definite) covariance matrix. Suppose we want to estimate the variance of a function h of the estimator B. Keeping only the first two terms of the Taylor series, and using vector notation for the gradient, we can estimate $h(B)$ as

$$h(B) \approx h(\beta) + \nabla h(\beta)^T (B-\beta) ,$$

which implies the variance of $h(B)$ is approximately

$$\begin{aligned}
\mathrm{Var}\big(h(B)\big) &\approx \mathrm{Var}\Big(h(\beta) + \nabla h(\beta)^T (B-\beta)\Big) \\
&= \mathrm{Var}\Big(h(\beta) + \nabla h(\beta)^T B - \nabla h(\beta)^T \beta\Big) \\
&= \mathrm{Var}\Big(\nabla h(\beta)^T B\Big) \\
&= \nabla h(\beta)^T \ \mathrm{cov}(\beta)\ \nabla h(\beta) .
\end{aligned}$$

One can use the mean value theorem (for real-valued functions of many variables) to see that this does not rely on taking first order approximation.

The delta method therefore implies that

$$\sqrt{n}\Big(h(B) - h(\beta)\Big) \xrightarrow{D} N\Big(0, \nabla h(\beta)^T\, \Sigma\ \nabla h(\beta)\Big) .$$

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