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THE BETA GENERALIZED INVERTED EXPONENTIAL DISTRIBUTION WITH REAL DATA APPLICATIONS

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Abstract:

- The four-parameter beta generalized inverted exponential distribution is considered in this article. Various properties of the model with graphs of the density function are investigated. Moreover, the maximum likelihood method of estimation is used for estimating the parameters of the model under complete samples. An asymptotic Fisher information matrix of the estimators is found. Additionally, confidence interval estimates of the parameters are obtained. The performances of findings of the article are shown by demonstrating various numerical illustrations through Monte Carlo simulation studies. Finally, applications on real data-sets are provided.

Key-Words:

- *beta generalized inverted exponential distribution; Fisher information matrix; goodness-of-fit test; maximum likelihood estimator; Monte Carlo simulation.*

AMS Subject Classification:

- 62-07, 62E20, 62F10, 62F25, 62N02.

1. INTRODUCTION

The generalized inverted exponential distribution (GIED) was introduced first by Abouammoh and Alshingiti (2009). It is a generalized form of the inverted exponential distribution (IED). IED has been studied by Keller and Kamath (1982) and Duran and Lewis (1989). GIED has good statistical and reliability properties. It fits various shapes of failure rates.

The probability density function (pdf) of a two-parameter GIED is given by

$$f(x) = \left(\frac{\alpha\lambda}{x^2}\right)\exp\left(\frac{-\lambda}{x}\right)\left[1-\exp\left(\frac{-\lambda}{x}\right)\right]^{\alpha-1}, \qquad x>0, \quad \alpha,\lambda>0\,, \tag{1.1}$$

and the cumulative distribution function (cdf) is given by

$$F(x) = 1-\left[1-\exp\left(\frac{-\lambda}{x}\right)\right]^{\alpha}, \qquad x>0, \quad \alpha,\lambda>0\,. \tag{1.2}$$

In the last few years, new classes of distributions have been found by extending certain distributions such that these new classes will have more applications in reliability, biology and other fields.

Let $G(t)$ be a cdf of a random variable T, such that

$$F(t) \;=\; \frac{1}{B(a,b)}\int_0^{G(t)} \varpi^{a-1}(1-\varpi)^{b-1}d\varpi\,, \tag{1.3}$$

where $a>0$, $b>0$, and $B(a,b)=\int_0^1 \varpi^{a-1}(1-\varpi)^{b-1}d\varpi$ is the beta function. The skewness of the distribution is controlled by the two parameters a and b. The cdf $G(t)$ could be any arbitrary distribution, and, consequently, F is named the beta G distribution. The previous formula in (1.3) was defined by Eugene *et al.* (2002) as a class of generalized distributions.

The beta normal distribution (BND) was introduced by Eugene *et al.* (2002). They used the cdf $G(t)$ of the normal distribution in (1.3) and derived some moments of the distribution. Expanding on this work, Gupta and Nadarajah (2004) established more general moments of BND. Based on the cdf $G(t)$ of the Gumbel distribution, Nadarajah and Kotz (2004) presented the beta Gumbel distribution and provided closed form expressions for the moments and the asymptotic distribution of the extreme order statistics and obtained the maximum likelihood estimators (MLE) of the parameters. Further, by using the cdf $G(t)$ of the exponential distribution, Nadarajah and Kotz (2005) considered the beta exponential distribution. They studied the first four cumulants, the moment generating function, and the extreme order statistics and found the MLE. Furthermore, Lee *et al.* (2007) considered the beta Weibull distribution and studied applications based on censored data.

Recently, Barreto-Souza *et al.* (2010) proposed the beta generalized exponential distribution by taking G(t) in (1.3) to be the cdf of the exponentiated exponential distribution and discussed the MLE of its parameters. Additionally, Nassar and Nada (2011) presented several properties of the beta generalized Pareto distribution. They estimated the distribution's parameters using the MLE. An application on actual tax revenue data was investigated. Paranaiba *et al.* (2011) discussed the beta Burr XII distribution. Mahmoudi (2011) presented the beta generalized Pareto distribution. Cordeiro and Lemonte (2011) investigated the beta Laplace distribution. Zea *et al.* (2012) studied statistical properties and inference of the beta exponentiated Pareto distribution (BEPD). They provided an application of the BEPD to remission times of bladder cancer. Leão *et al.* (2013) studied the beta inverse Rayleigh distribution. They provided various properties, including the quantile function, moments, mean deviations, Bonferroni and Lorenz curves, Rényi and Shannon entropies and order statistics, as well as the MLE. Baharith *et al.* (2014) discussed properties, the MLE and the Fisher information matrix for the beta generalized inverse Weibull distribution.

In this paper, a new beta distribution is introduced by taking $G(\cdot)$ to be the GIED, and we refer to it as the beta generalized inverted exponential distribution (BGIED). In Section 2, the BGIED is defined. Statistical properties of the model are derived in Section 3. Maximum likelihood estimators of the parameters are derived in Section 4. In Section 5, the asymptotic Fisher information matrix is investigated. Additionally, interval estimates of the parameters are found using the maximum likelihood method in Section 6. Section 7 explains the simulation studies that illustrate the theoretical results. Finally, Section 8 provides applications to real data-sets. Various conclusions are addressed in Section 9.

2. BETA GENERALIZED INVERTED EXPONENTIAL DISTRIBUTION

In this section, we introduce the four-parameter beta generalized inverted exponential distribution (BGIED) by assuming $G(x)$ to be the cdf of the generalized inverted exponential distribution (GIED). Substituting (1.2), the cdf of GIED, into (1.3), the cdf of the BGIED is obtained in the following form

$$F(x) = \frac{1}{B(a,b)} \int_0^{1-\left[1-\exp\left(\frac{-\lambda}{x}\right)\right]^{\alpha}} \varpi^{a-1}(1-\varpi)^{b-1} d\varpi \,, \qquad x>0, \quad a,b,\alpha \text{ and } \lambda>0 \,. \tag{2.1}$$

The pdf of the BGIED takes the form

$$f(x) = \frac{\alpha\lambda \exp\left(\frac{-\lambda}{x}\right)}{x^2 B(a,b)} \left(1-\left[1-\exp\left(\frac{-\lambda}{x}\right)\right]^{\alpha}\right)^{a-1} \left[1-\exp\left(\frac{-\lambda}{x}\right)\right]^{\alpha b-1}, \qquad x>0, \quad a,b,\alpha \text{ and } \lambda>0 \,. \tag{2.2}$$

For a positive real value $a > 0$, (2.2) can be rewritten as an infinite power series in the form

$$f(x) = \frac{\alpha\lambda \exp\left(\frac{-\lambda}{x}\right)}{x^2 B(a,b)} \sum_{k=0}^{\infty} \frac{(-1)^k \Gamma(a)}{k!\Gamma(a-k)} \left[1 - \exp\left(\frac{-\lambda}{x}\right)\right]^{\alpha(b+k)-1}, \quad x > 0, \quad a, b, \alpha, \text{ and } \lambda > 0 \,. \tag{2.3}$$

From (2.3), the corresponding cdf can be written as follows

$$F(x) = \frac{1}{B(a,b)} \sum_{k=0}^{\infty} \frac{(-1)^{k+1}}{a(b+k)B(a-k,k+1)} \left[1 - \exp\left(\frac{-\lambda}{x}\right)\right]^{\alpha(b+k)}, \quad x > 0, \quad a, b, \alpha \text{ and } \lambda > 0 \,. \tag{2.4}$$

The GIED is a special case of (2.2) when $a = b = 1$. Therefore, we can assume all of the properties of the GIED that were investigated by Abouammoh and Alshingiti (2009) still hold. Additionally, when $\alpha = 1$ in (2.2), the BIED is obtained, which is related to the BGIWD when the shape parameters are equal to one and has been discussed by Baharith *et al.* (2014).

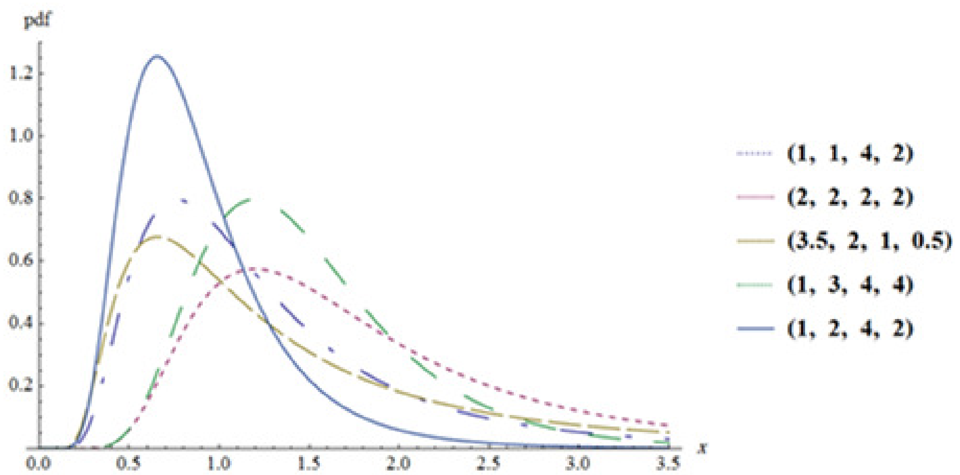

Figure 1: The pdf curves of the BGIED with (a, b, α, λ).

3. STATISTICAL PROPERTIES

3.1. The reliability and hazard functions

The reliability function is the probability of no failure occurring before time t. Alternately, the hazard function is the instantaneous rate of failure at a given time. These two functions are very important properties of a lifetime distribution.

The reliability function of the BGIED is given by

$$R(x) \;=\; 1-\frac{1}{B(a,b)}\sum_{k=0}^{\infty}\frac{(-1)^{k+1}}{a(b+k)B(a-k,k+1)}\left[1-\exp\left(\frac{-\lambda}{x}\right)\right]^{\alpha(b+k)}, \tag{3.1}$$

$$x>0,\quad a,b,\alpha \text{ and } \lambda>0\,,$$

and the corresponding hazard function of the BGIED can be written as

$$h(x)=\frac{\frac{\alpha\lambda\exp\left(\frac{-\lambda}{x}\right)}{x^2B(a,b)}\sum_{k=0}^{\infty}\frac{(-1)^k\Gamma(a)}{k!\Gamma(a-k)}\left[1-\exp\left(\frac{-\lambda}{x}\right)\right]^{\alpha(b+k)-1}}{1-\frac{1}{B(a,b)}\sum_{k=0}^{\infty}\frac{(-1)^{k+1}}{a(b+k)B(a-k,k+1)}\left[1-\exp\left(\frac{-\lambda}{x}\right)\right]^{\alpha(b+k)}}, \tag{3.2}$$

$$x>0,\quad a,b,\alpha \text{ and } \lambda>0\,.$$

Figure 2 shows different choices for the parameters of the BGIED. Additionally, it is shown from Figure 3 that the hazard function of the BGIED has an upside down bathtub shape. As is shown, the hazard function increases and then decreases.

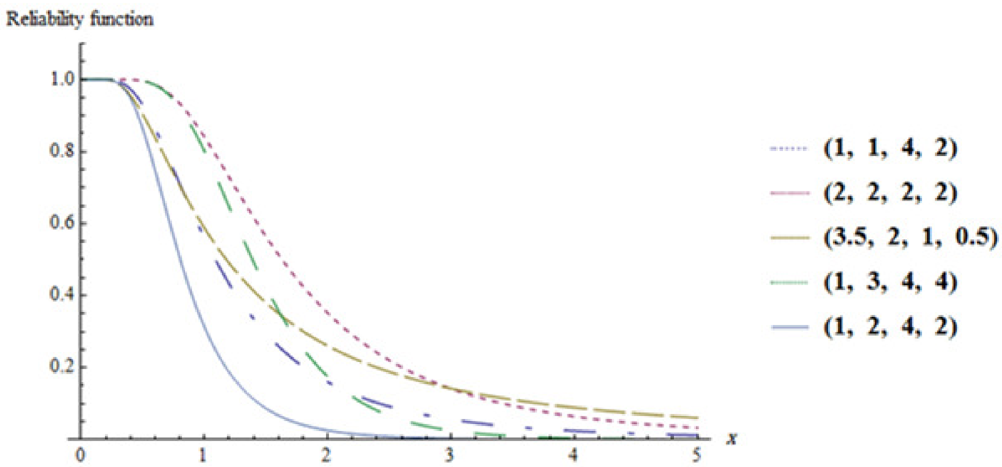

Figure 2: The reliability curves of the BGIED with (a, b, α, λ).

The upside down bathtub hazard function indicates that the risk of failing decreases as soon as the item has passed a specific time, during which it may have experienced some type of stress. Thus, the BGIED shows good statistical behavior based on these two functions.

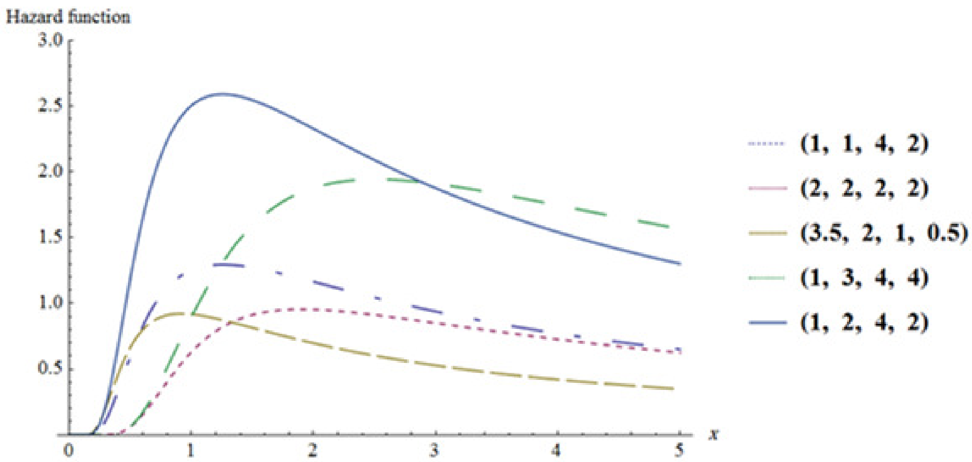

Figure 3: The hazard curves of the BGIED with (a, b, α, λ).

3.2. Moments and various measures

The r^{th} moment about the origin, $\mu'_r = E(X^r)$ of a BGIED with pdf (2.2) in the non-closed form is

$$\mu'_r = \int_0^\infty x^r \frac{\alpha\lambda \exp\left(\frac{-\lambda}{x}\right)}{x^2 B(a,b)} \left(1 - \left[1 - \exp\left(\frac{-\lambda}{x}\right)\right]^\alpha\right)^{a-1} \left[1 - \exp\left(\frac{-\lambda}{x}\right)\right]^{\alpha b - 1} dx\,, \quad r = 1, 2, ...$$

that is, for $k \geq r$, μ'_r takes the closed form

$$\text{(3.3)} \qquad \mu'_r = \frac{\lambda^r}{B(a,b)} \sum_{k=0}^{\infty} \sum_{j=0}^{\infty} \frac{(-1)^{k+j}(j+1)^{r-1}}{a(b+k)B(a-k,k+1)B(j+1,\alpha(b+k)-j)} \times \left\{ \sum_{i=0}^{\infty} \frac{(-1)^i}{i!(i-r+1)} + E_r(1) \right\},$$

where $B(a,b)$ is the beta function, and $E_n(z)$ is called the exponential integral function (Abramowitz and Stegun (1972)), which is defined as

$$\text{(3.4)} \qquad E_n(z) = \int_1^\infty \frac{\exp(-zt)}{t^n}\, dt\,.$$

Substituting $r = 1$ in (3.3), we obtain the mean of the BGIED as follows

$$\text{(3.5)} \qquad \mu = \frac{\lambda}{B(a,b)} \sum_{k=1}^{\infty} \sum_{j=0}^{\infty} \frac{(-1)^{k+j}}{a(b+k)B(a-k,k+1)B(j+1,\alpha(b+k)-j)} \times \left\{ \sum_{i=1}^{\infty} \frac{(-1)^i}{i^2(i-1)!} + E_1(1) \right\},$$

where $E_1(1) = 0.577216$ is Euler's constant.

Additionally, the variance of the BGIED can be found from

$$\text{(3.6)}\quad \operatorname{Var}(x) = \frac{\lambda^2}{B(a,b)} \sum_{k=2}^{\infty} \sum_{j=0}^{\infty} \frac{(-1)^{(k+j)}(j+1)}{a(b+k)B(a-k,k+1)B(j+1,\alpha(b+k)-j)} \times \left\{ \sum_{i=0}^{\infty} \frac{(-1)^i}{i!(i-1)} + E_2(1) \right\} - \mu^2 .$$

3.3. Quantile function and various related measures

The quantile function of the BGIED corresponding to (2.2) is

$$\text{(3.7)}\qquad q(u) = -\lambda \Big/ \log\left\{1 - \left[1 - I_u^{-1}(a,b)\right]^{\frac{1}{\alpha}}\right\}, \qquad 0 < u < 1 ,$$

where $I_u^{-1}(a,b)$ is the inverse of the incomplete beta function with parameters a and b, such that

$$I_u(a,b) = \frac{1}{B(a,b)} \int_0^u \varpi^{a-1}(1-\varpi)^{b-1} d\varpi ,$$

The above form of $q(u)$ allows us to derive the following forms of statistical measures for the BGIED:

1. The first quartile $Q1$, the second quartile $Q2$ (median), and the third quartile $Q3$ of the BGIED correspond to the values $u = 0.25$, 0.50, and 0.75, respectively
2. The median (m), also, can be found using (2.4) such that $\left|1-\exp\left(\frac{-\lambda}{m}\right)\right| < 1$, for $a = 1$, and then

$$\text{(3.8)}\qquad m = \frac{-\lambda}{\log\left[1 - (-0.5)^{\frac{1}{\alpha b}}\right]} .$$

3. The skewness and kurtosis can be calculated by using the following relations, respectively:

Bowley's skewness is based on quartiles; Kenney and Keeping (1962) calculated it as follows

$$\text{(3.9)}\qquad \upsilon_3 = \frac{Q3 - 2\,Q2 + Q1}{Q3 - Q1} ,$$

Moors' kurtosis (Moors (1988)) is based on octiles via the form

$$\text{(3.10)}\qquad \upsilon_4 = \frac{q(7/8) - q(5/8) - q(3/8) + q(1/8)}{q(6/8) - q(2/8)} ,$$

where $q(\cdot)$ represents the quantile function defined in (3.7).

When $a = b = 1$ in (2.3), (3.3) and (3.7) give the moments and the quantile of GIED, and, when $a = b = \alpha = 1$ in (2.3), (3.3) and (3.7) give the moments and the quantile of IED. Therefore, all measures above are satisfied for GIED when $a = b = 1$, and for IED when $a = b = \alpha = 1$.

3.4. The mean deviation

Let X be a BGIED random variable with mean $\mu = E(X)$ and median m. In this subsection, the mean deviation from the mean and the mean deviation from the median are derived.

3.4.1. The mean deviation from the mean can be found from the following theorem:

Theorem 1. *The mean deviation from the mean of the BGIED is in the form*

$$\begin{aligned} E(|X-\mu|) &= \frac{2}{B(a,b)} \sum_{k=0}^{\infty} \sum_{j=0}^{\infty} \frac{(-1)^{k+1+j}}{a(b+k)B(a-k,k+1)\left[\alpha(b+k)+1\right]} \\ &\times \frac{\mu \exp\left(-j\lambda/\mu\right) - j\lambda\Gamma\left(0, j\lambda/\mu\right)}{B\left[j+1, \alpha(b+k)-j+1\right]} \,, \end{aligned}$$

where $\Gamma(a,z) = \int_z^{\infty} t^{a-1} \exp(-t)\, dt$.

Proof: The mean deviation from the mean can be defined as

$$\begin{aligned} \mathrm{E}(|X-\mu|) &= \int_0^{\infty} |X-\mu| \, f(x) \; dx \\ &= 2\int_0^{\mu} (X-\mu) \, f(x) \; dx \\ &= 2\mu \, F(\mu) - 2I(\mu) \,, \end{aligned}$$

where $I(z) = \int_0^z t\, dG(t)$, and $d\left[t.dG(t)\right] = G(t)\, dt + t\, dG(t)$.

Therefore, $\mathrm{E}(|X-\mu|) = 2\int_0^{\mu} F(x)\, dx$.

Using (2.4), and expanding the term $(1-\exp(-\lambda/x))^{\alpha(b+k)}$ we obtain

$$\begin{aligned} \mathrm{E}(|X-\mu|) &= \frac{2}{B(a,b)} \sum_{k=0}^{\infty} \sum_{j=0}^{\infty} \frac{(-1)^{(k+1+j)}}{a(b+k)B(a-k,k+1)\left[\alpha(b+k)+1\right]} \\ &\times \frac{1}{B(j+1, \alpha(b+k)-j+1)} \int_0^{\mu} \exp\left(-j\lambda/x\right) \, dx \,, \end{aligned}$$

where

$$\int_0^c \exp(-j\lambda/x)\,dx = c\exp(-j\lambda/c) - j\lambda\Gamma(0, j\lambda/c)\ . \tag{3.11}$$

Hence, the theorem is proved. □

3.4.2. The mean deviation from the median can be found from the following theorem:

Theorem 2. *The mean deviation from the median of the BGIED is in the form*

$$E(|X-m|) = \mu + \frac{2}{B(a,b)}\sum_{k=0}^{\infty}\sum_{j=0}^{\infty}\frac{(-1)^{(k+j)}j\lambda}{a(b+k)B(a-k,k+1)\left[\alpha(b+k)+1\right]} \times \frac{\Gamma(0,j\lambda/m)}{B(j+1,\alpha(b+k)-j+1)}\ , \qquad j\lambda>0,\quad m>0\ .$$

Proof: The mean deviation from the median can be defined as

$$\begin{aligned}
\mathrm{E}(|X-m|) &= \int_0^\infty |x-m| f(x)dx \\
&= 2\int_0^m (m-x)\,f(x)dx - \int_0^m (m-x)\,f(x)dx + \int_m^\infty (x-m)\,f(x)dx \\
&= 2\int_0^m (m-x)\,f(x)dx + \int_0^\infty (x-m)\,f(x)dx \qquad (3.12)\\
&= \mu - 2\left[mF(m) - \int_0^m F(x)dx\right] \\
&= \mu - m + 2\int_0^m F(x)dx\ .
\end{aligned}$$

Substituting (2.4) into (3.12) and using (3.11), we obtain

$$\begin{aligned}
\mathrm{E}(|X-m|) &= \mu - m \\
&\quad + \frac{2}{B(a,b)}\sum_{k=0}^{\infty}\sum_{j=0}^{\infty}\frac{(-1)^{k+j+1}\left[m\exp(-j\lambda/m) - j\lambda\Gamma(0,j\lambda/m)\right]}{a(b+k)B(a-k,k+1)\left[\alpha(b+k)+1\right]} \\
&\quad \times \frac{1}{B\left[j+1,\alpha(b+k)-j+1\right]} \\
&= \mu - m + 2mF(m) \\
&\quad + \frac{2}{B(a,b)}\sum_{k=0}^{\infty}\sum_{j=0}^{\infty}\frac{(-1)^{k+j}j\lambda\Gamma(0,j\lambda/m)}{a(b+k)B(a-k,k+1)\left[\alpha(b+k)+1\right]} \\
&\quad \times \frac{1}{B\left[j+1,\alpha(b+k)-j+1\right]}\ .
\end{aligned}$$

Hence, the theorem is proved. □

3.5. The mode

The mode for the BGIED can be found by differentiating $f(x)$ with respect to x; thus, (2.2) gives

$$
\begin{aligned}
f'(x) \;=\; & f(x)\left\{\frac{-2}{x}+\frac{\lambda}{x^2}-(\alpha b-1)\,[1-\exp(-\lambda/x)]^{-1}\,\frac{\lambda}{x^2}\exp(-\lambda/x)\right.\\
& +\;(a-1)\,[1-(1-\exp(-\lambda/x))^{\alpha}]^{-1}\\
& \left.\times\;\frac{\alpha\lambda}{x^2}\exp(-\lambda/x)\,(1-\exp(-\lambda/x))^{\alpha-1}\right\}.
\end{aligned}
\tag{3.13}
$$

By equating (3.13) with zero, we get

$$
\begin{aligned}
& 1-\frac{2x}{\lambda}+(\exp(-\lambda/x)-1)^{-1}\;\times\\
& \times\left\{\alpha\,(a-1)\left[(1-\exp(-\lambda/x))^{-\alpha}-1\right]^{-1}-(\alpha b-1)\right\} \;=\; 0\,.
\end{aligned}
\tag{3.14}
$$

Then, the mode of the BGIED can be found numerically by solving (3.14).

In Table 1, we present the values of the mean, standard deviation (SD), mode, median, skewness and kurtosis for different values of a, b, α and λ.

Table 1: The mean, SD, mode, median, skewness and kurtosis for different values of the parameters.

a	b	α	λ	mean	SD	mode	median	skewness	kurtosis
1	1	4	2	1.35919	1.04298	0.76393	1.08802	0.23016	0.66022
1	1	4	4	2.71838	2.08595	1.52787	2.17604	0.23016	0.66022
1	2	4	4	1.79735	0.88156	1.30871	1.60709	0.16158	0.44815
1	3	4	4	1.50143	0.61555	1.19585	1.38881	0.13109	0.35990
1	2	4	2	0.89867	0.44078	0.65435	0.80354	0.16158	0.44815
2	1	4	2	1.81971	1.24786	1.12748	1.50317	0.22047	0.63372
2	2	2	2	1.98654	1.39808	1.19879	1.62873	0.22323	0.63985
2	2	2	4	3.97308	2.79615	2.39758	3.25747	0.22323	0.63985
3.5	2	1	0.5	1.97910	302.436	0.65618	1.17708	0.32893	1.01650

The results of studying the behaviour of the BGIED are shown in Table 1 and Figure 1. We note that the distribution is unimodal and positively skewed. For fixed values of a, b and α, the kurtosis values remain constant; therefore, the mode, median and mean increase with the increase of λ. As we increase the value of $\alpha \geq 1$ and fix the other parameters, the kurtosis value increases and the mean decreases. It is noted that the distribution has a long right tail for fixed values of b, α and λ. Moreover, for fixed values of a, α and λ the kurtosis and the mean values decrease as we increase the value of b. Additionally, for different values of α and λ and fixed values of a and b, the skewness and the kurtosis values remain stable. Alternately, for fixed values of α and λ, the skewness and the kurtosis values decrease as we increase a and b. Furthermore, we found that our results for $a = b = 1$ are exactly the same as the results in Abouammoh and Alshingiti (2009).

4. MAXIMUM LIKELIHOOD ESTIMATORS

In this section, we examine estimation by maximum likelihood and inference for the BGIED. Let X_1, X_2, ..., X_n be a random sample from the BGIED with pdf and cdf given, respectively, by (2.2) and (2.4). The likelihood function in this case can be written as (Lawless (2003)):

$$L(\underline{\theta}|\underline{x}) = \prod_{i=1}^{n} f(x_i) , \tag{4.1}$$

where $f(\cdot)$ is given by (2.2) and $\underline{\theta} = (a, b, \alpha, \lambda)$.

The natural logarithm of the likelihood function (4.1) is given by

$$\ell = \log L(\underline{\theta}|\underline{x}) = \sum_{i=1}^{n} \log f(x_i) . \tag{4.2}$$

For the BGEID, we have have

$$\begin{aligned} \log L = {} & n \log(\alpha\lambda) - n \log B(a, b) + (\alpha b - 1) \sum_{i=1}^{n} \log\left(1 - \exp\left(\frac{-\lambda}{x_i}\right)\right) \\ & -\lambda \sum_{i=1}^{n} x_i^{-1} - 2 \sum_{i=1}^{n} \log(x_i) + (a-1) \sum_{i=1}^{n} \log\left[1 - \left(1 - \exp\left(\frac{-\lambda}{x_i}\right)\right)^{\alpha}\right] . \end{aligned} \tag{4.3}$$

Assuming that the parameters $\underline{\theta} = (a, b, \alpha, \lambda)$, are unknown, the likelihood equations are given for $\underline{\theta}$

$$l_j = \frac{\partial \log L}{\partial \theta_j} = \frac{1}{f(x_i)} \frac{\partial f(x_i)}{\partial \theta_j} = 0 , \qquad j = 1, 2, 3, 4 .$$

From (2.2), we have

$$\frac{\partial \log L}{\partial \alpha} = \frac{n}{\alpha} + \sum_{i=1}^{n} \log\left(1 - \exp\left(\frac{-\lambda}{x_i}\right)\right) \left\{ b - (a-1) \left[\left(1 - \exp\left(\frac{-\lambda}{x_i}\right)\right)^{-\alpha} - 1\right]^{-1} \right\} , \tag{4.4}$$

$$\frac{\partial \log L}{\partial \lambda} = \frac{n}{\lambda} + (\alpha b - 1)\sum_{i=1}^{n} x_i^{-1}\left(\exp\left(\frac{\lambda}{x_i}\right) - 1\right)^{-1} - \sum_{i=1}^{n} x_i^{-1}$$

$$-\alpha(a-1)\sum_{i=1}^{n} x_i^{-1}\left[\left(1 - \exp\left(\frac{-\lambda}{x_i}\right)\right)^{-\alpha} - 1\right]^{-1}\left(\exp\left(\frac{\lambda}{x_i}\right) - 1\right)^{-1}, \tag{4.5}$$

$$\frac{\partial \log L}{\partial a} = \frac{-n}{B(a,b)}\phi_1 + \sum_{i=1}^{n} \log\left[1 - \left(1 - \exp\left(\frac{-\lambda}{x_i}\right)\right)^{\alpha}\right],$$

$$\begin{aligned}\phi_1 &= \frac{\partial B(a,b)}{\partial a} = \frac{\Gamma(b)\left[\Gamma(a+b)\Gamma'(a) - \Gamma(a)\partial\Gamma(a+b)/\partial a\right]}{\left[\Gamma(a+b)\right]^2}\\ &= B(a,b)\left[\psi(a) - \psi(a+b)\right],\end{aligned}$$

where $\psi(z) = \frac{1}{\Gamma(z)}\frac{\partial\Gamma(z)}{\partial z} = \frac{\Gamma'(z)}{\Gamma(z)}$ is called the Psi function (Abramowitz and Stegun (1972)). Then,

$$\frac{\partial \log L}{\partial a} = -n\left[\psi(a) - \psi(a+b)\right] + \sum_{i=1}^{n} \log\left[1 - \left(1 - \exp\left(\frac{-\lambda}{x_i}\right)\right)^{\alpha}\right], \tag{4.6}$$

$$\frac{\partial \log L}{\partial b} = \frac{-n}{B(a,b)}\phi_2 + \alpha\sum_{i=1}^{n} \log\left(1 - \exp\left(\frac{-\lambda}{x_i}\right)\right),$$

$$\begin{aligned}\phi_2 &= \frac{\partial B(a,b)}{\partial b} = \frac{\Gamma(a)\left[\Gamma(a+b)\Gamma'(b) - \Gamma(b)\partial\Gamma(a+b)/\partial b\right]}{\left[\Gamma(a+b)\right]^2}\\ &= B(a,b)\left[\psi(b) - \psi(a+b)\right],\end{aligned}$$

$$\frac{\partial \log L}{\partial b} = -n\left[\psi(b) - \psi(a+b)\right] + \alpha\sum_{i=1}^{n} \log\left(1 - \exp\left(\frac{-\lambda}{x_i}\right)\right). \tag{4.7}$$

The solution of the four nonlinear likelihood equations via (4.4), (4.5), (4.6) and (4.7) yields the maximum likelihood estimates (MLEs) $\hat{\underline{\theta}} = (\hat{a}, \hat{b}, \hat{\alpha}, \hat{\lambda})$ of $\underline{\theta} = (a, b, \alpha, \lambda)$. These equations are in implicit form, so they may be solved using numerical iteration, such as the Newton–Raphson method via Mathematica 9.0.

5. ASYMPTOTIC VARIANCES AND COVARIANCES OF ESTIMATES

The asymptotic variances of maximum likelihood estimates are given by the elements of the inverse of the Fisher information matrix $I_{ij}\left(\underline{\theta}\right) = E\left(-\frac{\partial^2 \ln L}{\partial\theta_i \partial\theta_j}\right)$. Unfortunately, the exact mathematical expressions for the above expectation are very difficult to obtain. Therefore, the observed Fisher information matrix is given by $I_{ij} = -\frac{\partial^2 \ln L}{\partial\theta_i \partial\theta_j}$ which is obtained by dropping the expectation on operation E (Cohen (1965)). The approximate (observed) asymptotic variance-covariance matrix F for the maximum likelihood estimates of the BGIED can be written as follows

$$F = [I_{ij}(\underline{\theta})] , \qquad i,j = 1,2,3,4 \quad \text{and} \quad \underline{\theta} = (a,b,\alpha,\lambda) .$$

The second partial derivatives of the maximum likelihood function for the BGIED are given as the following

$$\begin{aligned}\frac{\partial^2 \log L}{\partial\lambda^2} &= \frac{-n}{\lambda^2} - (\alpha b - 1)\sum_{i=1}^{n} x_i^{-2}\left(\exp\left(\frac{\lambda}{x_i}\right)-1\right)^{-1}\left[1+\left(\exp\left(\frac{\lambda}{x_i}\right)-1\right)^{-1}\right] \\ &\quad - \alpha(a-1)\sum_{i=1}^{n} x_i^{-2}\left[\left(1-\exp\left(\frac{-\lambda}{x_i}\right)\right)^{-\alpha}-1\right]^{-1}\left(\exp\left(\frac{\lambda}{x_i}\right)-1\right)^{-1} \\ &\quad \times\left\{-1+\left(\exp\left(\frac{\lambda}{x_i}\right)-1\right)^{-1}\left((\alpha-1)+\alpha\left[\left(1-\exp\left(-\frac{\lambda}{x_i}\right)\right)^{-\alpha}-1\right]^{-1}\right)\right\},\end{aligned}$$

$$\begin{aligned}\frac{\partial^2 \log L}{\partial\alpha^2} &= \frac{-n}{\alpha^2} - (a-1)\sum_{i=1}^{n}\left[\log\left(1-\exp\left(\frac{-\lambda}{x_i}\right)\right)\right]^2\left[\left(1-\exp\left(\frac{-\lambda}{x_i}\right)\right)^{-\alpha}-1\right]^{-1} \\ &\quad \times\left\{1+\left[\left(1-\exp\left(\frac{-\lambda}{x_i}\right)\right)^{-\alpha}-1\right]^{-1}\right\},\end{aligned}$$

$$\frac{\partial^2 \log L}{\partial a^2} = -n\left[\psi'(a)-\psi'(a+b)\right] ,$$

where

$$\begin{aligned}\psi'(z) &= \frac{\partial\psi(z)}{\partial z} , \\ \frac{\partial^2 \log L}{\partial b^2} &= -n\left[\psi'(b)-\psi'(a+b)\right] , \\ \frac{\partial^2 \log L}{\partial a\,\partial b} &= n\,\psi'(a+b) ,\end{aligned}$$

$$\begin{aligned}\frac{\partial^2 \log L}{\partial\lambda\,\partial\alpha} &= \sum_{i=1}^{n} x_i^{-1}\left(\exp\left(\frac{\lambda}{x_i}\right)-1\right)^{-1}\left\{b-(a-1)\left[\left(1-\exp\left(\frac{-\lambda}{x_i}\right)\right)^{-\alpha}-1\right]^{-1}\right. \\ &\quad \left.\times\left[1+\frac{\alpha\log\left(1-\exp\left(\frac{-\lambda}{x_i}\right)\right)}{\left[1-\left(1-\exp\left(\frac{-\lambda}{x_i}\right)\right)^{\alpha}\right]}\right]\right\},\end{aligned}$$

$$\frac{\partial^2 \log L}{\partial\lambda\,\partial a} = -\alpha\sum_{i=1}^{n} x_i^{-1}\left[\left(1-\exp\left(\frac{-\lambda}{x_i}\right)\right)^{-\alpha}-1\right]^{-1}\left(\exp\left(\frac{\lambda}{x_i}\right)-1\right)^{-1},$$

$$\frac{\partial^2 \log L}{\partial\lambda\,\partial b} = \alpha \sum_{i=1}^{n} x_i^{-1} \left(\exp\left(\frac{\lambda}{x_i}\right) - 1\right)^{-1},$$

$$\frac{\partial^2 \log L}{\partial\alpha\,\partial a} = -\sum_{i=1}^{n} \log\left(1 - \exp\left(\frac{-\lambda}{x_i}\right)\right) \left[\left(1 - \exp\left(\frac{-\lambda}{x_i}\right)\right)^{-\alpha} - 1\right]^{-1},$$

$$\frac{\partial^2 \log L}{\partial\alpha\,\partial b} = \sum_{i=1}^{n} \log\left(1 - \exp\left(\frac{-\lambda}{x_i}\right)\right).$$

Consequently, the maximum likelihood estimators of a, b, α and λ and have an asymptotic variance-covariance matrix defined by inverting the Fisher information matrix F and by substituting $\hat{a}$ for a, $\hat{b}$ for b, $\hat{\alpha}$ for α and $\hat{\lambda}$ for λ.

6. INTERVAL ESTIMATES

If $L_\theta = L_\theta\,(y_1, ..., y_n)$ and $U_\theta = U_\theta\,(y_1, ..., y_n)$ are functions of the sample data $y_1, ..., y_n$ then a confidence interval for a population parameter θ is given by

$$P\Big(L_\theta \le \theta \le U_\theta\Big) = \gamma\,,$$

where L_θ and U_θ are the lower and upper confidence limits that enclose θ with probability γ. The interval $[L_\theta, U_\theta]$ is called a $100\gamma\%$ confidence interval for θ.

For large sample sizes (Bain and Engelhardt (1992)), the maximum likelihood estimates, under appropriate regularity conditions, are consistent and asymptotically normally distributed. Therefore, the approximate $100\gamma\%$ confidence limits for the maximum likelihood estimate $\hat{\theta}$ of a population parameter θ can be constructed, such that

$$P\left(-z \le \frac{\hat{\theta} - \theta}{\sigma(\hat{\theta})} \le z\right) = \gamma\,, \tag{6.1}$$

where z is the $\frac{100(1+\gamma)}{2}$ standard normal percentile. Therefore, the approximate $100\gamma\%$ confidence limits for a population parameter θ can be obtained such that

$$P\left(\hat{\theta} - z\sigma(\hat{\theta}) \le \theta \le \hat{\theta} + z\sigma(\hat{\theta})\right) = \gamma\,. \tag{6.2}$$

Then, the approximate confidence limits for a, b, α and λ will be constructed using (6.2) with a confidence level of 90%.

7. SIMULATION STUDIES

Simulation studies have been performed using Mathematica 9.0 to illustrate the theoretical results of the estimation problem. The performance of the resulting estimators of the parameters has been considered in terms of their absolute relative bias (ARBias) and mean square error (MSE), where

$$\text{ARBias}\left(\hat{\theta}\right) = \left|\frac{\hat{\theta}-\theta}{\theta}\right| \qquad \text{and} \qquad \text{MSE}\left(\hat{\theta}\right) = \text{E}\left(\hat{\theta}-\theta\right)^2.$$

Furthermore, the asymptotic variance, covariance matrix and confidence intervals of the parameters are obtained. The algorithm for the simulation procedure is described below:

Step 1. 1000 random samples of sizes $n = 10(10)50, 100, 200$ and 300 were generated from the BGIED. The true parameter values are selected as ($a = 1,\ b = 2,\ \alpha = 4,\ \lambda = 2$).

Step 2. For each sample, the parameters of the distribution are estimated under the complete sample.

Step 3. The Newton–Raphson method is used for solving the four nonlinear likelihoods for α, λ, a and b given in (4.4), (4.5), (4.6) and (4.7), respectively.

Step 4. The ARBiase and MSE of the estimators for the four parameters for all sample sizes are tabulated.

Step 5. For large sample sizes $n = 100, 200$ and 300, the Fisher information matrix of the estimators are computed using the equations presented in Section 5.

Step 6. By inverting the Fisher information matrix that was computed in Step 5, the asymptotic variances and covariances of the estimators are found.

Step 7. Based on the values of the asymptotic variances and covariances matrix that were found in Step 6 and on Eq. (6.2), the approximate confidence limits at 90% for the parameters are computed.

Simulation results are summarized in Tables 2, 3 and 4. Table 2 gives the ARBias and MSE of the estimators. The asymptotic variances and covariances matrix of the estimators for complete samples of size $n = 100, 200$ and 300 and true parameter values ($a = 1$, $b = 2$, $\alpha = 4$, $\lambda = 2$) are displayed in Table 3. The approximate confidence limits at 90% for the parameters are presented in Table 4.

Table 2: The ARBias and MSE of the parameters $\underline{\theta} = (a, b, \alpha, \lambda)$.

n	$\hat{a}$		$\hat{b}$		$\hat{\alpha}$		$\hat{\lambda}$	
	ARBias	MSE	ARBias	MSE	ARBias	MSE	ARBias	MSE
10	2.77997	57.22180	1.23642	39.93170	0.34929	57.85210	0.04445	4.49668
20	1.57487	18.92080	0.75400	18.13730	0.01542	23.75870	0.02126	3.08791
30	1.17703	8.04064	0.59348	10.72860	0.13843	12.21520	0.03641	2.48411
40	1.01435	7.53433	0.32958	4.15377	0.25344	7.47854	0.12458	1.52254
50	0.81716	5.63524	0.35502	5.58289	0.23793	8.44674	0.08708	1.65372
100	0.34507	2.01241	0.13037	1.57136	0.37237	5.26341	0.15154	0.73170
200	0.09587	0.78450	0.04185	0.88988	0.45129	4.69708	0.18296	0.46773
300	0.05260	0.56916	0.04620	0.88359	0.46429	4.67339	0.18716	0.31603

Table 3: Asymptotic variances and covariances of estimates for complete samples.

n	Parameters	$\hat{a}$	$\hat{b}$	$\hat{\alpha}$	$\hat{\lambda}$
100	a	0.00860	0.00389	−0.00021	−0.00359
	b	0.00389	0.05006	−0.00513	0.00867
	α	−0.00021	−0.00513	0.04464	0.01414
	λ	−0.00359	0.00867	0.01414	0.01308
200	a	0.00673	0.00327	−0.00018	−0.00411
	b	0.00327	0.03238	−0.00883	0.00346
	α	−0.00018	−0.00883	0.02928	0.00772
	λ	−0.00411	0.00346	0.00772	0.00951
300	a	0.00630	0.00320	−0.00075	−0.00445
	b	0.00320	0.02563	−0.00722	0.00197
	α	−0.00075	−0.00722	0.02245	0.00664
	λ	−0.00445	0.00197	0.00664	0.00888

Table 4: Confidence bounds of the estimates at a confidence level of 0.90.

n	Parameters	Estimated mean	Lower bound	Upper bound	Width
100	a	1.34507	1.19294	1.49719	0.30424
	b	2.26075	1.89380	2.62770	0.73390
	α	2.51052	2.16399	2.85704	0.69304
	λ	1.69693	1.50931	1.88455	0.375241
200	a	1.09587	0.96132	1.23041	0.26909
	b	2.08370	1.78856	2.37883	0.59027
	α	2.19482	1.91416	2.47549	0.56132
	λ	1.63408	1.47409	1.79408	0.31999
300	a	1.05260	0.92241	1.18279	0.26038
	b	2.09239	1.82981	2.35498	0.52517
	α	2.14283	1.89709	2.38857	0.49147
	λ	1.62569	1.47108	1.78030	0.30922

From these tables, the following observations can be made on the performance of the parameter estimation of the BGIED:

1. As the sample size increases, the MSEs of the estimated parameters decrease. This indicates that the maximum likelihood estimates provide asymptotically normally distributed and consistent estimators for the parameters (see Table 2).

2. Although the estimators of a and b are consistent according to the ARBias, it is noted that the estimators of α and λ are not consistent. Table 2 shows that, for large sample sizes (n = 100, 200 and 300), the ARBiases are increased, which indicates that the estimates of α and λ are not consistent.

3. The asymptotic variances of the estimators decrease when the sample size is increasing (see Table 3).

4. The interval of the estimators decreases when the sample size is increasing (see Table 4).

5. The interval estimations of all parameters were reasonable except the interval estimate of α. The estimated intervals at a confidence level of 0.90 for $n = 100, 200$ and 300 did not cover the real value of α.

We conclude from the previous points that the MLE of the parameters is a good estimator for a, b and λ.

8. APPLICATIONS

In this section, two sets of data are presented to demonstrate the utility of using the BGIED. These two sets were investigated by Abouammoh and Alshingiti (2009). The GIED was fitted to both of these sets.

8.1. The first data-set

The following data-set is presented in Lawless (2003). The data resulted from a test on the endurance of deep groove ball bearings. The data are as follows:

17.88, 28.92, 33.0, 41.52, 42.12, 45.60, 48.40, 51.84, 51.96, 54.12, 55.56, 67.80, 68.64, 68.64, 68.88, 84.12, 93.12, 98.64, 105.12, 105.84, 127.92, 128.04, 173.4 .

Descriptive statistics of these data are tabulated in Table 5.

Table 5: Descriptive statistics for the ball bearing data.

Measure	Value	Measure	Value
n	23	Minimum	17.880
Maximum	173.400	Mean	72.22
$Q1$	45.600	$Q3$	98.640
Median	67.800	Mean deviation	29.429
Variance	1405.580	SD	37.491
Skewness	0.941	Kurtosis	3.486

We apply the Kolmogorov–Smirnov (K-S) statistic to verify which distribution better fits these data. The K-S test statistic is described in detail in D'Agostino and Stephens (1986). In general, the smaller the value of K-S is, the better the fit to the data is. All graphs and computations presented to analyse the data were carried out by Mathematica 9.0. The model selection was carried out using the AIC (Akaike information criterion), the BIC (Bayesian information criterion) and the CAIC (consistent Akaike information criterion):

$$AIC = -2\,l(\hat{\underline{\theta}}) + 2p\,,$$

$$BIC = -2\,l(\hat{\underline{\theta}}) + p\log n$$

and

$$CAIC = -2\,l(\hat{\underline{\theta}}) + \frac{2pn}{n-p-1}\,,$$

where $l(\hat{\underline{\theta}})$ denotes the log likelihood function evaluated at the maximum likelihood estimates, p is the number of parameters and n is the sample size. Table 6 lists the values of the K-S statistic and of $-2l(\hat{\underline{\theta}})$. The K-S goodness-of-fit test of the BGIED as well as the GIED are the best among all models; accordingly, the BGIED model can be used to analyse the ball bearing data. Table 7 provides the MLEs with corresponding standard errors (SEs) of the model parameters.

Table 6: Goodness-of-fit measures and K-S statistics for the ball bearing data.

Model	BGIED	GIED	IED	BIED
K-S statistics	0.097	0.091	0.306	0.104
$-2l(\hat{\underline{\theta}})$	227.723	227.098	243.452	228.288

Table 7: MLEs of the model parameters, the corresponding SEs and the statistics of the AIC, BIC and CAIC for the ball bearing data.

Model	Method	Estimates				Statistics		
		$\hat{a}$	$\hat{b}$	$\hat{\alpha}$	$\hat{\lambda}$	AIC	BIC	CAIC
BGIED	MLE	20.615	9.427	0.532	7.161	235.723	240.265	237.946
	SE	0.125	0.279	8.052	0.262			
GIED	MLE			5.307	129.996	231.098	233.369	231.698
	SE			0.188	0.015			
IED	MLE				55.055	245.452	246.587	245.642
	SE				0.018			
BIED	MLE	15.858	3.692		11.858	234.288	237.695	235.552
	SE	0.112	0.508		0.161			

8.2. The second data-set

The data that is studied in this section was provided by Ed Fuller of the NICT Ceramics Division in December 1993. It contains polished window strength data. Fuller *et al.* (1994) described the use of this set to predict the lifetime for a glass airplane window. The data are as follows:

18.83, 20.8, 21.657, 23.03, 23.23, 24.05, 24.321, 25.5, 25.52, 25.8, 26.96, 26.77, 26.78, 27.05, 27.67, 29.9, 31.11, 33.2, 33.73, 33.76, 33.89, 34.76, 35.75, 35.91, 36.98, 37.08, 37.09, 39.58, 44.045, 45.29, 45.381 .

Descriptive statistics of the window strength data are tabulated in Table 8.

Table 8: Descriptive statistics for the window strength data.

Measure	Value	Measure	Value
n	31	Minimum	18.830
Maximum	45.381	Mean	30.820
$Q1$	25.500	Q3	35.910
Median	29.900	Mean deviation	6.145
Variance	52.539	SD	7.248
Skewness	0.403	Kurtosis	2.290

The K-S goodness-of-fit test of the BGIED as well as the BIED are the best among all models; therefore, the BGIED model can be used to study the window strength data. Table 10 presents the MLEs with corresponding SEs of the model parameters.

Table 9: Goodness-of-fit measures and K-S statistics for the window strength data.

Model	BGIED	GIED	IED	BIED
K-S statistics	0.133	0.137	0.474	0.130
$-2l(\hat{\underline{\theta}})$	208.207	208.454	274.523	208.105

Table 10: MLEs of the model parameters, the corresponding SEs and the statistics of the AIC, BIC and CAIC for the window strength data.

Model	Method	Estimates				Statistics		
		$\hat{a}$	$\hat{b}$	$\hat{\alpha}$	$\hat{\lambda}$	AIC	BIC	CAIC
BGIED	MLE	27.850	7.354	1.978	17.601	216.207	221.943	217.745
	SE	0.088	0.341	2.401	0.247			
GIED	MLE			90.855	148.412	212.454	215.322	212.883
	SE			0.011	0.029			
IED	MLE				29.215	276.523	277.957	276.661
	SE				0.034			
BIED	MLE	14.506	20.169		26.053	214.105	218.407	214.994
	SE	0.205	0.146		0.166			

Finally, we conclude the following from studying the AIC, BIC and CAIC statistics of the two previous data-sets.

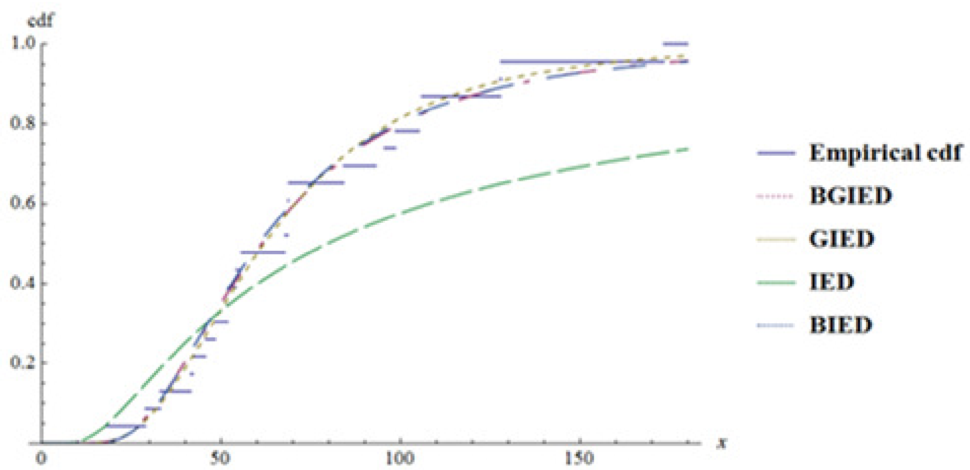

Figure 4: The empirical distribution and estimated cdf of the models for the ball bearing data.

It is noted that the GIED has a smaller value compared with the values of other models for the two data-sets. The BIED and BGIED follow next. That indicates that the GIED seems to be a very competitive model for these data. Because the values of the AIC, BIC and CAIC are approximately equivalent for the GIED, BIED and BGIED, the BGIED can thus be a good alternative model for these data, as can the GIED. Alternately, the IED presents the worst fit for the second dataset. Figure 4 shows the empirical distribution and estimated cdf of the models for the ball bearing data. Figure 5 shows the empirical distribution and estimated cdf of the models for the window strength data.

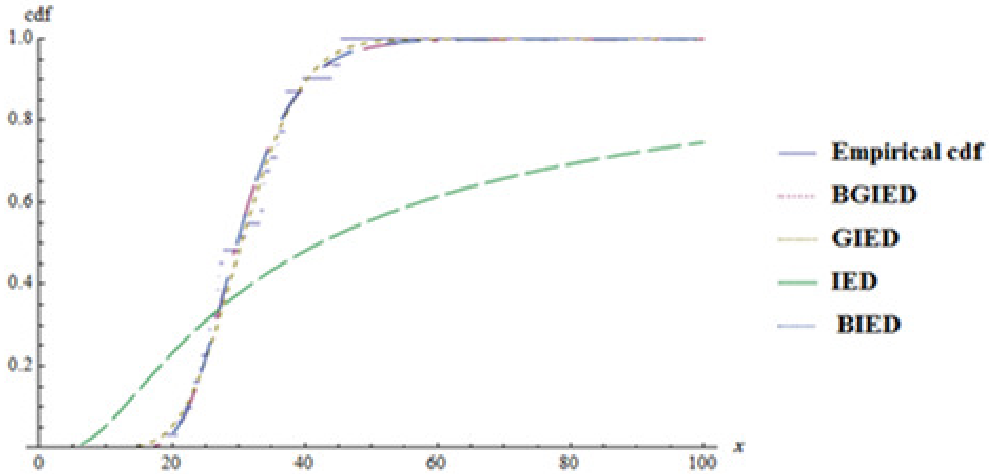

Figure 5: The empirical distribution and estimated cdf of the models for the window strength data.

9. CONCLUDING REMARKS

In this study, the four-parameter beta generalized inverted exponential distribution (BGIED) is proposed. BGIED generalizes the generalized inverted exponential distribution discussed by Abouammoh and Alshingiti (2009). Additionally, the BGIED represents a generalization of the inverted exponential distribution (IED). IED has been considered by Keller and Kamath (1982) and Duran and Lewis (1989). Statistical properties of the BGIED are studied. Maximum likelihood estimators of the BGIED parameters are obtained. The information matrix and the asymptotic confidence bounds of the parameters are derived. Monte Carlo simulation studies are conducted under different sample sizes to study the theoretical performance of the MLE of the parameters. Two real data-sets are analysed, and the BGIED has provided a good fit for the data-sets.

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