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SOME CHARACTERIZATION RESULTS BASED ON DOUBLY TRUNCATED REVERSED RESIDUAL LIFETIME RANDOM VARIABLE

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Abstract:

- In view of the growing importance of the reversed residual lifetime in reliability analysis and stochastic modeling, in this paper, we try to study some of the reliability properties of reversed residual lifetime random variable based on doubly truncated data and complete the preceding results such as Ruiz and Navarro (1995, 1996), Navarro *et al.* (1998), Nanda *et al.* (2003), Nair and Sudheesh (2008) and Sudheesh and Nair (2010). The monotonicity properties of the doubly truncated reversed residual variance and its relations with doubly truncated reversed residual expected value and doubly truncated reversed residual coefficient of variation are discussed. Furthermore, an upper bound for it under some conditions is obtained. We also discuss and find the similar results and some characterizations for discrete random ageing, which are noticeable in comparing with continuous cases.

Key-Words:

- *doubly truncated reversed residual expected value; doubly truncated reversed residual coefficient of variation; doubly truncated reversed residual variance; generalized failure rate; reversed failure rate.*

AMS Subject Classification:

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1. INTRODUCTION

In the reliability literature and analyzing survival data for the components of a system or a device, there have been defined several measures for various conditions and situations of the system. Sometimes, we only have information about between two lifetime points, so studying the reliability measures under the condition of doubly truncated random variables, is necessary. If the random variable X denotes the lifetime of a unit, then the random variable ${}_xX_y = (y - X \mid x \leq X \leq y)$ is called the doubly truncated reversed residual lifetime and it measures the time elapsed since the failure of X given that the system has been working since time x and has failed sometime before y. Note that the well-known random variable, $X_y = (y - X \mid X \leq y)$, which some of researchers called it, "reversed residual lifetime (RRL)", "past time to failure", "inactivity time" or "idle time", is the special case of ${}_xX_y$ when $x = 0$.

The subject of doubly truncation of a lifetime random variable in reliability literature has been started by Navarro and Ruiz (1996) and Ruiz and Navarro (1995, 1996) that generalized the failure rate function for doubly truncated random variables. Later, Sankaran and Sunoj (2004) defined and obtained some properties of the expected value of the doubly truncated lifetime distributions. Recently, many authors such as Su and Huang (2000), Ahmad (2001), Betensky and Martin (2003), Navarro and Ruiz (2004), Bairamov and Gebizlioglu (2005), Poursaeed and Nematollahi (2008) and Sunoj *et al.* (2009), studied the properties of the conditional expectations of doubly truncated random variables in various areas like order statistics and k-out-of-n systems. Also, recently Khorashadizadeh *et al.* (2012) have studied the doubly truncated mean residual lifetime and the doubly truncated mean past to failure and also obtained some characterization results in both continuous and discrete cases.

In this paper, we study some reliability measures based on the doubly truncated reversed lifetime random variable, ${}_xX_y$, in both continuous and discrete lifetime distributions, which some of the results that achieved are not similar to each other. The relationship among doubly truncated reversed residual expected (mean) value ($dRRM$), doubly truncated reversed residual variance ($dRRV$) and doubly truncated reversed residual coefficient of variation ($dRRCV$) are obtained. Also, their monotonicity and the associated ageing classes of distributions are discussed. Some characterization results of the class of the increasing $dRRV$ are presented and an upper bound for $dRRV$ under some conditions is obtained. Furthermore, we characterize the discrete distribution based on doubly truncated covariance and obtained some results for binomial, Poisson and negative binomial distributions.

2. CONTINUOUS DOUBLY TRUNCATED REVERSED RESIDUAL LIFETIME

Let X be a non-negative continuous random variable with cumulative distribution function (cdf), $F(x)$ and probability density function (pdf), $f(x)$. Navarro and Ruiz (1996) defined and studied the generalized failure rate (GFR) to the doubly truncated continuous random variables by

$$h_1(x,y) = \lim_{h\to 0^+}\left[\frac{P\big(x\le X\le x+h \mid x\le X\le y\big)}{h}\right] = \frac{f(x)}{F(y)-F(x)}$$

and

$$h_2(x,y) = \lim_{h\to 0^-}\left[\frac{P\big(y+h\le X\le y \mid x\le X\le y\big)}{h}\right] = \frac{f(y)}{F(y)-F(x)}\ ,$$

for $(x,y)\in D=\{(x,y);\, F(x)<F(y)\}$. Note that the special cases, $h_1(x,\infty)=\frac{f(x)}{1-F(x)}$ is the failure rate and $h_2(0,y)=\frac{f(y)}{F(y)}$ is the reversed failure rate. Navarro and Ruiz (1996) have shown that GFR determines the distribution uniquely.

We denote the continuous doubly truncated reversed residual expected value by $dRRM$ and the continuous doubly truncated reversed residual variance by $dRRV$ and define them as

$$\tilde\mu(x,y) = E({}_xX_y) = E\big(y-X \,|\, x\le X\le y\big)$$

and

$$\tilde\sigma^2(x,y) = \mathrm{Var}({}_xX_y) = \mathrm{Var}\big(y-X \,|\, x\le X\le y\big) = \mathrm{Var}\big(X \,|\, x\le X\le y\big)\ ,$$

respectively, such that $E(X^2)<\infty$, $(x,y)\in D$ and $\tilde\mu(x,x)=\tilde\sigma^2(x,x)=0$. Ruiz and Navarro (1995, 1996) and Navarro *et al.* (1998) have shown that $m(x,y)=E(X \,|\, x\le X\le y)$ determines $F(x)$ uniquely. So, this is also true for $\tilde\mu(x,y)=y-m(x,y)$.

The $dRRM$ can be rewritten as

$$\begin{aligned}\tilde\mu(x,y) &= E\big(y-X \,|\, x\le X\le y\big)\\ &= \frac{(x-y)\,F(x)+\int_x^y F(t)\,dt}{F(y)-F(x)}\ .\end{aligned} \tag{2.1}$$

So, we have

$$\frac{\partial}{\partial y}\tilde\mu(x,y) = \frac{\big[F(y)-F(x)\big]^2 - f(y)\big[(x-y)\,F(x)+\int_x^y F(t)\,dt\big]}{\big[F(y)-F(x)\big]^2}\ .$$

By using the above equation, the $\tilde\mu(x,y)$ determine the general failure rate, $h_2(x,y)$, via relation

$$h_2(x,y) = \frac{1-\frac{\partial}{\partial y}\tilde\mu(x,y)}{\tilde\mu(x,y)}\ , \qquad (x,y)\in D\ .$$

Furthermore, using part by part integration method, we can see that $\tilde{\sigma}^2(x,y)$ and $\tilde{\mu}(x,y)$ are related via the following equation:

$$\tilde{\sigma}^2(x,y) = \frac{(y^2-x^2)\,F(x) - 2\int_x^y tF(t)\,dt}{F(y)-F(x)} + 2y\,\tilde{\mu}(x,y) - \tilde{\mu}^2(x,y)\,. \tag{2.2}$$

This equation will be useful in proving various other relationships. We have the following definitions of ageing classes related to the $\tilde{\mu}(x,y)$ and $\tilde{\sigma}^2(x,y)$.

Definition 2.1. A random variable X is said to be

- **(i)** increasing in doubly truncated reversed residual expected value ($IdRRM$) if for any $(x,y)\in D$, $\tilde{\mu}(x,y)$ is increasing in y,
- **(ii)** increasing in doubly truncated reversed residual variance ($IdRRV$) if for any $(x,y)\in D$, $\tilde{\sigma}^2(x,y)$ is increasing in y.

The dual classes are defined similarly. For the random variable X_y, Nanda *et al.* (2003) showed that the class of decreasing RRM is empty. Thus, in the next theorem, we answer the natural question that whether the classes of decreasing doubly truncated reversed residual expected value ($DdRRM$) and decreasing doubly truncated reversed residual variance ($DdRRV$) of life distributions are null or not.

Theorem 2.1.

I. *There exist no non-negative random variable that has $DdRRM$ property.*

II. *There exist no non-negative random variable that has $DdRRV$ property.*

Proof: The two part can be proved by assuming the opposite. Suppose that $\tilde{\mu}(x,y)$ is decreasing in y. From (2.1), we have

$$0 \le \tilde{\mu}(x,y) \le y-x\ , \qquad \forall (x,y)\in D\ ,$$

and also

$$\lim_{y\to x} \tilde{\mu}(x,y) = 0\ .$$

Thus, if $\tilde{\mu}(x,y)$ is decreasing in y, then $\tilde{\mu}(x,y) \le \tilde{\mu}(x,x) = 0$, for all $(x,y)\in D$, which is contradict the fact that $\tilde{\mu}(x,y)$ cannot be negative or identically zero. Similarly, for part II., on contrary, suppose that $\tilde{\sigma}^2(x,y)$ is decreasing in y. For all $(x,y)\in D$, we have

$$0 \le \tilde{\sigma}^2(x,y) \le (y^2-x^2)$$

and

$$\lim_{y\to x} \tilde{\sigma}^2(x,y) = 0\ .$$

Thus, if $\tilde{\sigma}^2(x,y)$ is decreasing in y, then $\tilde{\sigma}^2(x,y) \le \tilde{\sigma}^2(x,x) = 0$, for all $(x,y)\in D$, which is contradict the fact that variance cannot be negative or identically zero. $\square$

In the next theorem, we obtain an upper bound for $\tilde{\sigma}^2(x,y)$, when X has the $IdRRM$ property.

Theorem 2.2. *If the non-negative continuous random variable X, has the $IdRRM$ property, then,*

$$\tilde{\sigma}^2(x,y) < \tilde{\mu}^2(x,y)\ , \qquad (x,y) \in D\ . \tag{2.3}$$

Proof: According to (2.1), we have

$$\begin{aligned}\int_x^y \big[F(t)-F(x)\big]\,\tilde{\mu}(x,t)\,dt \;&=\; \int_x^y \left[\int_x^t F(z)\,dz + (x-t)\,F(x)\right]dt\\ &=\; y\int_x^y F(z)\,dz - \int_x^y z\,F(z)\,dz + \int_x^y (x-t)\,F(x)\,dt\ ,\end{aligned}$$

using (2.2), it implies that

$$\begin{aligned}&\frac{2}{F(y)-F(x)}\int_x^y \big[F(t)-F(x)\big]\,\tilde{\mu}(x,t)\,dt \;=\\ &\qquad=\; \frac{(y^2-x^2)\,F(x) - 2\int_x^y zF(z)\,dz}{F(y)-F(x)} + 2\,y\,\tilde{\mu}(x,y)\\ &\qquad=\; \tilde{\mu}^2(x,y) + \tilde{\sigma}^2(x,y)\ .\end{aligned}$$

So, we have

$$\begin{aligned}\tilde{\sigma}^2(x,y) - \tilde{\mu}^2(x,y) \;&=\; \frac{2}{F(y)-F(x)}\int_x^y \big[F(t)-F(x)\big]\,\big[\tilde{\mu}(x,t)-\tilde{\mu}(x,y)\big]\,dt\\ &<\; 0\ ,\end{aligned}$$

since $\tilde{\mu}(x,y)$ is increasing in y. This completes the proof. □

Now, we investigate the connection between $IdRRV$ and other classes of life distributions.

Theorem 2.3. *If $\tilde{\mu}(x,y)$ is increasing in y, then $\tilde{\sigma}^2(x,y)$ is increasing in y, i.e., the $IdRRM$ property is stronger than the $IdRRV$ property.*

Proof: The proof is trivial by using the following relation:

$$\frac{\partial}{\partial y}\tilde{\sigma}^2(x,y) \;=\; h_2(x,y)\,\big[\tilde{\mu}^2(x,y) - \tilde{\sigma}^2(x,y)\big]\ . \tag{2.4}$$ □

In special case, the Example 2.1 in Nanda *et al.* (2003) shows that the converse of the above theorem is not true.

Another reliability measure that has been recently considered and is related to the reversed residual variance and the reversed residual expected value, is the reversed residual coefficient of variation. So, in doubly truncated random variables we consider the doubly truncated reversed residual coefficient of variation ($dRRCV$) as

$$\tilde{\gamma}(x,y) = \frac{\tilde{\sigma}(x,y)}{\tilde{\mu}(x,y)}\,, \qquad (x,y) \in D\,. \tag{2.5}$$

The Eq. (2.4) can be written as

$$\frac{\partial}{\partial y}\tilde{\sigma}^2(x,y) \;=\; h_2(x,y)\,\tilde{\mu}^2(x,y)\left[1-\tilde{\gamma}^2(x,y)\right]\,, \tag{2.6}$$

so, $\tilde{\sigma}^2(x,y)$ is increasing in y according as $\tilde{\gamma}^2(x,y) \le 1$.

The next theorem characterizes the monotonic behavior of the variance of the random variable ${}_xX_y$. A similar result for the variance of ${}_xX$ has been given by Nanda *et al.* (2003).

Theorem 2.4. *The following statements are equivalent:*

(i) $\tilde{\sigma}^2(x,y)$ *is increasing in* y *for any fixed* x *such that* $(x,y) \in D$.

(ii) $\tilde{\gamma}^2(x,y) \le 1$ *for all* $(x,y) \in D$.

(iii) $\Phi(x,y) = \frac{E\left[(y-X)^2 \,|\, x \le X \le y\right]}{E\left[y-X \,|\, x \le X \le y\right]}$ *is increasing in* y *for any fixed* x *such that* $(x,y) \in D$.

Proof: Using (2.6) and (2.2) the results will follow. □

3. DISCRETE DOUBLY TRUNCATED REVERSED RESIDUAL LIFETIME

In reliability analysis, interests in discrete failure data came relatively late in comparison to its continuous analogue.

Suppose T be a non-negative discrete random variable with support $\{0,1,2,...\}$ and cdf, $F(t)$ and probability mass function (pmf), $p(t)$. Navarro and Ruiz (1996) defined the generalized failure rate (GFR) to the doubly truncated discrete random variables for all $(t,k) \in D^* = \{(t,k);\, F(t^-) < F(k)\}$ by

$$h_1(t,k) \;=\; \frac{p(t)}{F(k)-F(t-1)} \tag{3.1}$$

and

$$h_2(t,k) \;=\; \frac{p(k)}{F(k)-F(t-1)}\,. \tag{3.2}$$

Let ${}_tT_k = (k - T\,|\,t \le T \le k)$ be the reversed doubly truncated random variable in discrete lifetime distributions. So, the doubly truncated reversed residual expected value and reversed residual variance based on ${}_tT_k$ are as follow,

$$\begin{aligned}\tilde{\mu}_d(t,k) &= E({}_tT_k) = E(k - T\,|\,t \le T \le k)\;,\\ \tilde{\sigma}^2_d(t,k) &= \mathrm{Var}(k - T\,|\,t \le T \le k) = \mathrm{Var}(T\,|\,t \le T \le k)\;,\end{aligned}$$

respectively, where $(t,k) \in D^*$. The function $\tilde{\mu}_d(t,k)$ can be rewritten as follow,

$$\begin{aligned}\tilde{\mu}_d(t,k) &= E(k - T\,|\,t \le T \le k)\\ &= \frac{(t-k)\,F(t-1) + \sum_{i=t}^{k-1} F(i)}{F(k) - F(t-1)}\;.\end{aligned} \tag{3.3}$$

One can easily obtain that the doubly truncated reversed residual expected value can characterize the general failure rate $h_2(t,k)$ via the relation,

$$h_2(t,k) = 1 - \frac{\tilde{\mu}_d(t,k)}{1 + \tilde{\mu}_d(t,k-1)}\;, \qquad (t,k) \in D^*\;. \tag{3.4}$$

Khorashadizadeh *et al.* (2012) have shown that if T be discrete random variable with support $\{0,1,2,\ldots,m\}$ (m can be finite or infinite), then for a known t, $F(\cdot)$ can be uniquely recovered by $\tilde{\mu}_d(t,k)$ as follows:

$$F(k) = A_k + F(t-1)\,[1 - A_k]\;, \tag{3.5}$$

where $A_k = \prod_{i=k+1}^{m} \frac{\tilde{\mu}_d(t,i)}{1+\tilde{\mu}_d(t,i-1)}$ and $F(t-1) = \frac{A_{-1}}{A_{-1}-1}$.

The monotonic ageing classes of distributions, *IdRRM* and *IdRRV* in discrete cases can be defined similar to Definition 2.1. Based on the discrete random variable $T^*_k = (k - T\,|\,T < k)$, Goliforushani and Asadi (2008) showed that the class of decreasing reversed residual expected value is empty.

Theorem 3.1. *There is no non-degenerate discrete distribution that has DdRRM or DdRRV property.*

Proof: On contrary, suppose that $\tilde{\mu}_d(t,k)$ is decreasing in k, then for any fixed t, $\tilde{\mu}_d(t,k+1) \le \tilde{\mu}_d(t,k) \le \tilde{\mu}_d(t,t) = 0$, which is contradict the fact that $\tilde{\mu}_d(t,k) \ge 0$. Similar prove can be done for *DdRRV* property. $\square$

One can obtain that

$$\begin{aligned}\tilde{\sigma}^2_d(t,k) &= \frac{(k+t)\,(k-t+1)\,F(t-1) - 2\sum_{i=t}^{k} i\,F(i-1)}{F(k) - F(t-1)}\\ &\quad + (2k+1)\,\tilde{\mu}_d(t,k) - \tilde{\mu}^2_d(t,k)\;.\end{aligned} \tag{3.6}$$

In the next theorem, we obtain an upper bound for $\tilde{\sigma}_d^2(t,k)$, when $\tilde{\mu}_d(t,k)$ is increasing in t, which is not the same as that obtained in Theorem 2.2 for continuous case.

Theorem 3.2. *If the non-negative discrete random variable T, has $IdRRM$ property, then,*

$$\tilde{\sigma}_d^2(t,k) < \tilde{\mu}_d(t,k)\,[1+\tilde{\mu}_d(t,k)]\;, \qquad (t,k)\in D^*\;. \tag{3.7}$$

Proof: According to (3.3), we have

$$\begin{aligned} 2\sum_{i=t}^{k}\big[F(i)-F(t-1)\big]\,\tilde{\mu}_d(t,i) \;=\; &(k+t)(k-t+1)F(t-1) - 2\sum_{j=t}^{k} jF(j-1) \\ &+ (2k+2)\left[(t-k)\,F(t-1)+\sum_{j=t}^{k-1}F(j)\right]. \end{aligned} \tag{3.8}$$

Thus, dividing the both sides of (3.8) by $F(k)-F(t-1)$ and making use of (3.6), implies

$$\begin{aligned} \tilde{\sigma}_d^2(t,k)-\tilde{\mu}_d^2(t,k) \;&=\; \frac{2}{F(k)-F(t-1)}\sum_{i=t}^{k-1}\big[F(i)-F(t-1)\big]\big[\tilde{\mu}_d(i,k)-\tilde{\mu}_d(t,k)\big] \\ &\quad+\tilde{\mu}_d(t,k)\left[\frac{F(k)+F(t-1)}{F(k)-F(t-1)}\right] \\ &< \tilde{\mu}_d(t,k)\;, \end{aligned}$$

since $\tilde{\mu}_d(t,k)$ is increasing with respect k. Hence the required result is obtained. □

The connection between $IdRRV$ and other classes of distributions are also discussed for discrete case in the following theorem.

Theorem 3.3. *In discrete lifetime distributions, the $IdRRM$ property implies the $IdRRV$ property.*

Proof: Using (3.6), we have

$$\begin{aligned} \tilde{\sigma}_d^2(t,k)-\tilde{\sigma}_d^2(t,k-1) \;=\;& \frac{(k+t)\,(k-t+1)\,F(t-1)-2\sum_{i=t}^{k} i\,F(i-1)}{F(k)-F(t-1)} \\ &+\;(2k+1)\,\tilde{\mu}_d(t,k)-\tilde{\mu}_d^2(t,k) \\ &-\;\frac{(k+t-1)\,(k-t)\,F(t-1)-2\sum_{i=t}^{k-1} i\,F(i-1)}{F(k-1)-F(t-1)} \\ &-\;(2k-1)\,\tilde{\mu}_d(t,k-1)+\tilde{\mu}_d^2(t,k-1)\;. \end{aligned} \tag{3.9}$$

On the other hand, one can see that

$$\frac{(k+t)\,(k-t+1)\,F(t-1) - 2\sum_{i=t}^{k} i\,F(i-1)}{F(k)-F(t-1)} -$$
$$- \frac{(k+t-1)\,(k-t)\,F(t-1) - 2\sum_{i=t}^{k-1} i\,F(i-1)}{F(k-1)-F(t-1)} =$$
$$= \Big[(2k-1)\,\tilde{\mu}_d(t,k-1) - \tilde{\mu}_d^2(t,k-1) - \tilde{\sigma}_d^2(t,k-1) + 2k\Big]\,h_2(t,k) - 2k$$

and also

$$\tilde{\mu}_d(t,k) - \tilde{\mu}_d(t,k-1) \;=\; 1 - h_2(t,k)\left[\tilde{\mu}_d(t,k-1)+1\right] .$$

So, by using these two relations and summarizing the equations, we can write the Eq. (3.9) as

$$\tilde{\sigma}_d^2(t,k) - \tilde{\sigma}_d^2(t,k-1) \;= \tag{3.10}$$
$$=\; h_2(t,k)\Big[\tilde{\mu}_d(t,k)\,\tilde{\mu}_d(t,k-1) + \tilde{\mu}_d(t,k) - \tilde{\sigma}_d^2(t,k-1)\Big] .$$

Since, $\tilde{\mu}_d(t,k)$ is increasing in k,

$$\tilde{\sigma}_d^2(t,k) - \tilde{\sigma}_d^2(t,k-1) \;\ge\; h_2(t,k)\Big[\tilde{\mu}_d^2(t,k-1) + \tilde{\mu}_d(t,k-1) - \tilde{\sigma}_d^2(t,k-1)\Big] ,$$

so on using Theorem 3.2, we get the required results. □

The converse of the Theorem 3.3 is not true. The following counterexample shows that *IdRRV* property dose not imply the *IdRRM* property.

Example 3.1. Let T be a discrete random variable with cdf,

k	0	1	2	3	4	5
$F(k)$	0.0625	0.1046	0.1901	0.5561	0.875	1

One can see that in this distribution, $\tilde{\sigma}_d^2(0,k)$ is increasing in k, but $\tilde{\mu}_d(0,k)$ is not monotone.

We consider the discrete doubly truncated reversed residual coefficient of variation as

$$\tilde{\gamma}_d(t,k) \;=\; \frac{\tilde{\sigma}_d(t,k)}{\tilde{\mu}_d(t,k)} .$$

Another characterizations for the *IdRRV* and *IdRRM* classes of distributions based on $\tilde{\gamma}_d(t,k)$ are obtained in the next theorem.

Theorem 3.4. *For non-negative discrete random variable T, we have*

(i) *T has IdRRV property, if and only if,*

$$\tilde{\gamma}_d^2(t,k) \;\le\; \frac{\tilde{\mu}_d(t,k+1)}{\tilde{\mu}_d(t,k)}\left[1+\frac{1}{\tilde{\mu}_d(t,k)}\right].$$

(ii) *T has IdRRM property, if and only if,*

$$\tilde{\gamma}_d^2(t,k) \;\le\; 1+\frac{1}{\tilde{\mu}_d(t,k)}\;.$$

Proof: The statement (i) can be proved by using Eq. (3.10) and the statement (ii) can be proved by using Theorem 3.2. □

In the next theorem, we present a characterization via $\tilde{\sigma}_d^2(t,k)$ which is not quite similar to Theorem 2.4 in continuous case.

Theorem 3.5. *$\tilde{\sigma}_d^2(t,k)$ is increasing in k, if and only if,*

$$\frac{\tilde{\sigma}_d^2(t,k-1)}{\tilde{\mu}_d(t,k)\,\tilde{\mu}_d^+(t,k)} \;\le 1\;,$$

where $\tilde{\mu}_d^+(t,k) = E(k-T\,|\,t\le T<k)$.

Proof: Using (3.10) and $\tilde{\mu}_d^+(t,k)=\tilde{\mu}_d(t,k-1)+1$ the required result is obtained. □

4. CHARACTERIZATIONS OF SOME DISCRETE LIFETIME DISTRIBUTIONS

In this section, we characterize discrete distributions based on the doubly truncated random variables. Nair and Sudheesh (2008) and Sudheesh and Nair (2010) have presented some characterization results with their applications for discrete distributions based on one way truncated random variable. In the following theorems, we extend their results for doubly truncated random variables, which are more general and applicable. Since sometimes, the available information is in the specific interval period of times.

Let $c(\cdot)$ be any real valued function, so that for any $(t_1,t_2)\in D^*$,

$$m_c(t_1,t_2) \;=\; E\big(c(T)\,|\,t_1\le T\le t_2\big) \;=\; \frac{\sum_{i=t_1}^{t_2} c(i)\,p(i)}{F(t_2)-F(t_1-1)} \tag{4.1}$$

is the conditional expected value of doubly (interval) truncated random variable.

Theorem 4.1. *Let T be a non-negative discrete random variable with pmf, $p(t)$, and cdf, $F(t)$. Also suppose that for any real valued function $c(\cdot)$, $\mu = E(c(T))$ and $\sigma^2 = \mathrm{Var}(c(T))$, then for $(t_1, t_2) \in D^*$, T follows the family of distributions satisfying*

$$\frac{p(t+1)}{p(t)} = \frac{\sigma\, g(t)}{\sigma\, g(t+1) - \mu + c(t+1)}\,, \qquad t = 0, 1, 2, \ldots\,, \tag{4.2}$$

with $\sigma\, g(t) = \sum_{i=0}^{t} \frac{p(i)}{p(t)} \left[\mu - c(i)\right]$, *if and only if,*

$$m_c(t_1, t_2) = \mu + \sigma\, g(t_1 - 1)\, \frac{h_1(t_1 - 1, t_2)}{1 - h_1(t_1 - 1, t_2)} - \sigma\, g(t_2)\, h_2(t_1, t_2)\,. \tag{4.3}$$

Proof: Suppose (4.2) holds, then,

$$c(t)\, p(t) = \mu p(t) + \sigma p(t-1)\, g(t-1) - \sigma p(t)\, g(t)\,. \tag{4.4}$$

Summation the both side of (4.4) from t_1 to t_2 leads to

$$\sum_{i=t_1}^{t_2} \big[c(i) - \mu\big]\, p(i) = \sigma \big[p(t_1 - 1)\, g(t_1 - 1) - p(t_2)\, g(t_2)\big]\,. \tag{4.5}$$

Dividing the both sides of (4.5) by $F(t_2) - F(t_1 - 1)$ and using (3.1), (3.2) and (4.1), we get (4.3) and *vise versa*. □

Remark 4.1. It can be seen that, in special cases, when $t_1 \to 0$ or $t_2 \to \infty$, Theorem 4.1 is identical with that of Nair and Sudheesh (2008) and Sudheesh and Nair (2010).

In the next theorem, we characterize the family of the form (4.2) based on doubly truncated conditional covariance and expected value.

Theorem 4.2. *The distribution function of the non-negative discrete random variable T, belongs to the family of the form (4.2), if and only if, for all non-negative integer values $(t_1, t_2) \in D^*$,*

$$\begin{aligned} \mathrm{Cov}\big(s(T), c(T) \,|\, t_1 \le T \le t_2\big) &= \sigma E\big(\Delta s(T).g(T) \,|\, t_1 \le T \le t_2\big) \\ &\quad + \big[\mu - m_c(t_1, t_2)\big]\big[m_s(t_1, t_2) - s(t_2 + 1)\big] \\ &\quad - \sigma\, g(t_1 - 1)\, \frac{h_1(t_1 - 1, t_2)}{1 - h_1(t_1 - 1, t_2)} \big[s(t_2 + 1) - s(t_1)\big]\,, \end{aligned} \tag{4.6}$$

where $c(\cdot)$ and $s(\cdot)$ are any real valued functions such that $E(s^2(T)) < \infty$, $E(\Delta s(T)\,.\,g(T)) < \infty$, $\Delta s(T) \neq 0$ and $m_s(t_1, t_2) = E(s(T) \,|\, t_1 \le T \le t_2)$.

Proof: First, we know that

$$E\big(s(T)\,(c(T)-\mu)\,|\,t_1 \le T \le t_2\big) \;= \tag{4.7}$$

$$\begin{aligned}
&= \frac{1}{F(t_2)-F(t_1\!-\!1)} \sum_{i=t_1}^{t_2} s(i)\,\big[c(i)-\mu\big]\,p(i) \\
&= \frac{1}{F(t_2)-F(t_1\!-\!1)} \sum_{j=0}^{t_2-t_1} s(t_1+j)\left[\sum_{i=t_1+j}^{t_2}\big(c(i)-\mu\big)p(i) - \sum_{i=t_1+j+1}^{t_2}\big(c(i)-\mu\big)\,p(i)\right] \\
&= \frac{1}{F(t_2)-F(t_1\!-\!1)}\left[\sum_{j=t_1}^{t_2}\Delta s(j)\sum_{i=j+1}^{t_2}\big(c(i)-\mu\big)\,p(i)\right] + s(t_1)\,\big(m_c(t_1,t_2)-\mu\big)\,.
\end{aligned}$$

Now, suppose that T has a distribution of form (4.2), hence, on using

$$\sum_{i=j+1}^{t_2}\big(c(i)-\mu\big)\,p(i) \;=\; \sigma\big[p(j)\,g(j)-p(t_2)\,g(t_2)\big]\,,$$

we have

$$E\big(s(T)\,(c(T)-\mu)\,|\,t_1 \le T \le t_2\big) \;= \tag{4.8}$$

$$\begin{aligned}
&= \frac{1}{F(t_2)-F(t_1-1)}\left[\sigma\sum_{j=t_1}^{t_2}\Delta s(j)\,g(j)\,p(j) - \sigma\,g(t_2)\,p(t_2)\sum_{j=t_1}^{t_2}\Delta s(j)\right] \\
&\quad + s(t_1)\,\big(m_c(t_1,t_2)-\mu\big) \\
&= \sigma\,E\big(\Delta s(T)\,.\,g(T)\,|\,t_1 \le T \le t_2\big) \;+\; \sigma\,s(t_1)\,g(t_1-1)\,\frac{h_1(t_1-1,t_2)}{1-h_1(t_1-1,t_2)} \\
&\quad + s(t_2+1)\left[m_c(t_1,t_2)-\mu-\sigma\,g(t_1-1)\,\frac{h_1(t_1-1,t_2)}{1-h_1(t_1-1,t_2)}\right],
\end{aligned}$$

or

$$\begin{aligned}
&E\big(s(T)\,.\,c(T)\,|\,t_1 \le T \le t_2\big) \;= \\
&\quad = \sigma\,E\big(\Delta s(T)\,.\,g(T)\,|\,t_1 \le T \le t_2\big) \;+\; s(t_2+1)\,\big(m_c(t_1,t_2)-\mu\big) \\
&\qquad - \sigma\,g(t_1-1)\,\frac{h_1(t_1-1,t_2)}{1-h_1(t_1-1,t_2)}\,\big[s(t_2+1)-s(t_1)\big] \;+\; \mu\,m_s(t_1,t_2)\,,
\end{aligned}$$

which easily leads to (4.6).

Conversely, let (4.6) is true, then, comparing (4.7) and (4.8) implies

$$\begin{aligned}
&\sum_{j=t_1}^{t_2}\Delta s(j)\sum_{i=j+1}^{t_2}\big(c(i)-\mu\big)\,p(i) \;= \\
&\qquad = \sigma\sum_{j=t_1}^{t_2}\Delta s(j)\,g(j)\,p(j) - \sigma\,g(t_2)\,p(t_2)\sum_{j=t_1}^{t_2}\Delta s(j)\,.
\end{aligned} \tag{4.9}$$

Changing t_1 to t_1-1 and subtracting from (4.9) leads to (4.5), which is equivalent to the distribution with the form (4.2). $\square$

Remark 4.2. In special case of Theorem 4.2, when $s(T) = c(T) = T$, we have

$$\begin{aligned} \tilde{\sigma}^2_d(t_1,t_2) &= \mathrm{Var}\big(T \,|\, t_1 \le T \le t_2\big) \\ &= \sigma^* E\big(g(T) \,|\, t_1 \le T \le t_2\big) + \big[\mu^* - m(t_1,t_2)\big]\,\big[m(t_1,t_2) - t_2 - 1\big] \\ &\quad - \sigma^* g(t_1-1)\,\frac{h_1(t_1-1,t_2)}{1-h_1(t_1-1,t_2)}\,[t_2 - t_1 + 1]\,, \end{aligned} \tag{4.10}$$

where $m(t_1,t_2) = E(T \,|\, t_1 \le T \le t_2)$ is the doubly truncated expected time to failure function and $\mu^* = E(T)$ and $\sigma^{2*} = \mathrm{Var}(T)$.

In the table in the following page, we illustrate the results of Remark 4.2 in some distributions.

Remark 4.3. It should be noted that similar results and definitions in discrete case, can be verified by using the doubly truncated reversed random variables ${}_{t^+}T_k = (k - T \,|\, t < T \le k)$, ${}_{t}T_{k^-} = (k - T \,|\, t \le T < k)$ or ${}_{t^+}T_{k^-} = (k - T \,|\, t < T < k)$.

5. SUMMARY AND CONCLUSIONS

In this paper, we obtain some reliability properties of the reversed residual lifetime via doubly truncation. Also, their similarities and differences are compared in both discrete and continuous lifetime distributions and the following partial chain is obtained.

$$h_2(a,b) \text{ is decreasing in } b \implies IdRRM \implies IdRRV\,.$$

Also, some characterization results are obtained in discrete distributions via conditional covariance and variance.

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Distribution	$p(t)$	$\sigma^* g(t)$	$\mathrm{Var}\big(T \mid t_1 \le T \le t_2\big)$
Binomial	$\binom{n}{t} p^t (1-p)^{n-t}$ $t = 0, 1, ..., n$ $0 \le p \le 1$	$(n-t)\,p$	$p\left[m(t_1,t_2)\,(n-1) - nt_2 - \dfrac{h_1(t_1-1,t_2)}{1-h_1(t_1-1,t_2)}\,(t_2-t_1+1)\,(n-t_1+1)\right] -$ $-\,m(t_1,t_2)\,\big(m(t_1,t_2)-t_2-1\big)$
Poisson	$\dfrac{e^{-\theta}\,\theta^t}{t!}$ $t = 0, 1, ...$ $\theta \ge 0$	θ	$\theta\left[m(t_1,t_2) - \dfrac{t_2}{1-h_1(t_1-1,t_2)} + (t_1-1)\,\dfrac{h_1(t_1-1,t_2)}{1-h_1(t_1-1,t_2)}\right] -$ $-\,m(t_1,t_2)(m(t_1,t_2)-t_2-1)$
Neg. binomial	$\binom{k+t-1}{k-1}\theta^k\,(1-\theta)^t$ $t = 0, 1, 2, ...$ $0 \le \theta \le 1$	$\dfrac{1-\theta}{\theta}\,(t+k)$	$\dfrac{1}{\theta}\left[(t_1+k-1)\,(t_2-t_1+1)\,(\theta-1)\,\dfrac{h_1(t_1-1,t_2)}{1-h_1(t_1-1,t_2)} + \big(m(t_1,t_2)-t_2\big)\,k + m(t_1,t_2)\right] -$ $-\,(m(t_1,t_2)+k)(m(t_1,t_2)-t_2)$
Geometric	$\theta\,(1-\theta)^t$ $t = 0, 1, 2, ...$ $0 \le \theta \le 1$	$\dfrac{1-\theta}{\theta}\,(t+1)$	$\dfrac{t_1(t_2-t_1+1)}{1-(1-\theta)^{t_2-t_1+1}} + \dfrac{2\,m(t_1,t_2)-t_2}{\theta}\,\big(m(t_1,t_2)+1\big)\,\big(m(t_1,t_2)-t_2\big)$

REFERENCES

[1] Ahmad, A. A. (2001). Moments of order statistics from doubly truncated continuous distributions and characterizations, *Statist.*, **35**(4), 479–494.

[2] Bairamov, I. and Gebizlioglu, O. (2005). *On the characterizations of distributions through the properties of conditional expectations of order statistics*, In "Advances on Models, Characterizations and Applications" (N. Balakrishnan, I. Bairamov and O.L. Gebizlioglu, Eds.), Boca Raton, FL: Chapman and Hall / CRC Press.

[3] Betensky, R. A. and Martin, E. C. (2003). Commentary: Failure-rate functions for doubly truncated random variables, *IEEE Trans. on Reliab.*, **52**(1), 7–8.

[4] Golifroushani, S. and Asadi, M. (2008). On the discrete mean past lifetime, *Metrika*, **68**, 209–217.

[5] Khorashadizadeh, M.; Rezaei Roknabadi, A. H. and Mohtashami Borzadaran, G. R. (2012). Characterizations of lifetime distributions based on doubly truncated mean residual life and mean past to failure, *Commun. Statist. Theor. Meth.*, **41**(6), 1105–1115.

[6] Nair, U. N. and Sudheesh, K. K. (2008). Some results on lower variance bounds useful in reliability modeling and estimation, *Ann. Inst. Stat. Math.*, **60**, 591–603.

[7] Nanda, A. K.; Singh, H.; Misra, N. and Paul, P. (2003). Reliability properties of reversed residual lifetime, *Commun. Statist. Theor. Meth.*, **32**(10), 2031–2042.

[8] Navarro, J. and Ruiz, J. M. (1996). Failure-Rate Function for Doubly-Truncated Random Variables, *IEEE Trans. Reliab.*, **45**(4), 685–690.

[9] Navarro, J. and Ruiz, J. M. (2004). Characterization from relationships between failure rate functions and conditional moments, *Commun. Statist. Theor. Meth.*, **33**(12), 3159–3171.

[10] Navarro, J.; Ruiz, J. M. and Zoroa, N. (1998). A unified approach to characterization problems using conditional expectations, *J. Statist. Plann. Infer.*, **69**, 193–207.

[11] Poursaeed, M. H. and Nematollahi, A. R. (2008). On the mean past and the mean residual life under double monitoring, *Commun. Statist. Theor. Meth.*, **37**(7), 1119–1133.

[12] Ruiz, J. M. and Navarro, J. (1995). Characterization of discrete distributions using expectated values, *Statistical Papers*, **36**, 237–252.

[13] Ruiz, J. M. and Navarro, J. (1996). Characterization based on conditional expectations of the doubled truncated distribution, *Ann. Ins. Statist. Math.*, **48**(3), 563–572.

[14] Sankaran, P. G. and Sunoj, S. M. (2004). Identification of models using failure rate and mean residual life of doubly truncated random variables, *Statist. Pap.*, **45**, 97–109.

[15] Su, J. C. and Huang, W. J. (2000). Characterizations based on conditional expectations, *Statist. Pap.*, **41**, 423–435.

[16] Sudheesh, K. K. and Nair, N. U. (2010). Characterization of discrete distributions by conditional variance, *Metron*, **LXVIII**(1), 77–85.

[17] Sunoj, S. M.; Sankaran, P. G. and Maya, S. S. (2009). Characterizations of life distributions using conditional expectations of doubly (interval) truncated random variables, *Commun. Statist. Theor. Meth.*, **38**(9), 1441–1452.