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THE SHORTEST CLOPPER–PEARSON RANDOMIZED CONFIDENCE INTERVAL FOR BINOMIAL PROBABILITY

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Abstract:

- Zieliński (2010) showed the existence of the shortest Clopper–Pearson confidence interval for binomial probability. The method of obtaining such an interval was presented as well. Unfortunately, the confidence interval obtained has one disadvantage: it does not keep the prescribed confidence level. In this paper, a small modification is introduced, after which the resulting shortest confidence interval does not have the above mentioned disadvantage.

Key-Words:

- *binomial proportion; confidence interval; shortest confidence intervals.*

AMS Subject Classification:

- 62F25.

1. INTRODUCTION

The problem of estimating the probability of success in a binomial model has a very long history as well as very wide applications. Let us recall the definition of a confidence interval for probability of success $\theta \in (0,1)$ (see Cramér (1946), Lehmann (1959), Silvey (1970); for a general definition of confidence interval see Neyman (1934)):

A random interval $\big(\underline{\theta}(X), \overline{\theta}(X)\big)$ is called a *confidence interval for a parameter θ at the confidence level γ* if

$$P_\theta\Big\{\underline{\theta}(X) \le \theta \le \overline{\theta}(X)\Big\} \;\ge\; \gamma \qquad \text{for all} \quad \theta \in (0,1)\;.$$

Here X denotes the number of successes in a sample of size n.

It is easy to note that for any $d(n), g(n) > 0$ the interval $\big(\underline{\theta}(X) - d(n), \overline{\theta}(X) + g(n)\big)$ is of course also a confidence interval. So an additional criterion is needed for choosing a confidence interval. There are a lot different criterions. Clopper and Pearson, who proposed the first confidence interval for θ, took under consideration the equal risk of underestimation and overestimation. The problem of the shortest confidence intervals was seldom considered in the past (Crow 1956, Blyth and Hutchinson 1960, Blyth and Still 1983, Casella 1986). Zieliński (2010) proposed a simple method of obtaining the shortest confidence interval for θ. Unfortunately, the solution has a serious disadvantage: the proposed confidence interval does not keep the nominal confidence level. So, in what follows a slight modification is proposed. Namely, an auxiliary random variable $Y \in (0,1)$ is applied and the shortest confidence interval is constructed on the basis of $X+Y$. It appears that such a confidence interval does not have the above mentioned disadvantage.

2. THE CONFIDENCE INTERVAL

Consider the binomial statistical model

$$\Big(\big\{0,1,...,n\big\},\ \big\{Bin(n,\theta),\ 0<\theta<1\big\}\Big)\,,$$

where $Bin(n,\theta)$ denotes the binomial distribution with probability distribution function (pdf)

$$\binom{n}{k}\theta^k(1-\theta)^{n-k}\,, \qquad k = 0,1,...,n\;.$$

It is well known that

$$\sum_{k\le x}\binom{n}{k}\theta^k(1-\theta)^{n-k} \;=\; F\big(n-x,\, x+1;\, 1-\theta\big) \;=\; 1 - F\big(x+1,\, n-x;\, \theta\big)\,,$$

where $F(a,b;\cdot)$ is the cumulative distribution function (cdf) of the beta distribution with parameters (a,b).

Let X denote a binomial $Bin(n,\theta)$ random variable. A confidence interval for probability θ at the confidence level γ is of the form (Clopper and Pearson, 1934)

$$\Big(F^{-1}\big(X, n-X+1; \gamma_1\big);\ F^{-1}(X+1, n-X; \gamma_2)\Big)\,,$$

where $\gamma_1, \gamma_2 \in (0,1)$ are such that $\gamma_2 - \gamma_1 = \gamma$ and $F^{-1}(a,b;\alpha)$ is the α quantile of the beta distribution with parameters (a,b), i.e.

$$P_\theta\Big\{\theta \in \Big(F^{-1}\big(X, n-X+1; \gamma_1\big);\ F^{-1}\big(X+1, n-X; \gamma_2\big)\Big)\Big\} \ \geq\ \gamma\,, \qquad \forall\,\theta \in (0,1)\,.$$

For $X=0$ the left end is taken to be 0, and for $X=n$ the right end is taken to be 1.

Zieliński (2010) considered the length of the confidence interval when $X=x$ is observed:

$$d(\gamma_1, x) \ =\ F^{-1}\big(x+1, n-x; \gamma+\gamma_1\big) - F^{-1}\big(x, n-x+1; \gamma_1\big)\,.$$

Let x be given. The existence as well as the method of finding $0<\gamma_1^*<1-\gamma$ such that $d(\gamma_1^*, x)$ is minimal was shown. Examples of shortest confidence intervals (*left*, *right*) are given in Table 1.

Table 1: The shortest c.i. ($n=20$, $\gamma=0.95$).

x	γ_1^*	*left*	*right*
0	0.00000	0.00000	0.13911
1	0.00000	0.00000	0.21611
2	0.00125	0.00261	0.28393
3	0.00561	0.01839	0.34998
4	0.00966	0.04318	0.41249
5	0.01302	0.07344	0.47156
6	0.01587	0.10763	0.52766
7	0.01840	0.14496	0.58118
8	0.02071	0.18496	0.63234
9	0.02288	0.22733	0.68126
10	0.02500	0.27196	0.72804

By symmetry, for $x > n/2$ we have $\gamma_1^*(x) = (1-\gamma) - \gamma_1^*(n-x)$, $\mathit{left}(x) = 1 - \mathit{right}(n-x)$ and $\mathit{right}(x) = 1 - \mathit{left}(n-x)$. The confidence level of the shortest confidence interval for probability θ equals

$$\sum_{x=0}^{n} \binom{n}{x} \theta^x (1-\theta)^{n-x}\, \mathbf{1}(x,\theta)\,,$$

where

$$\mathbf{1}(x,\theta) \;=\; \begin{cases} 1 & \text{if } \theta \in \big(\mathit{left}(x), \mathit{right}(x)\big)\,, \\ 0 & \text{otherwise}\,. \end{cases}$$

For $n = 20$ and $\gamma = 0.95$ the confidence level is shown in Figure 1.

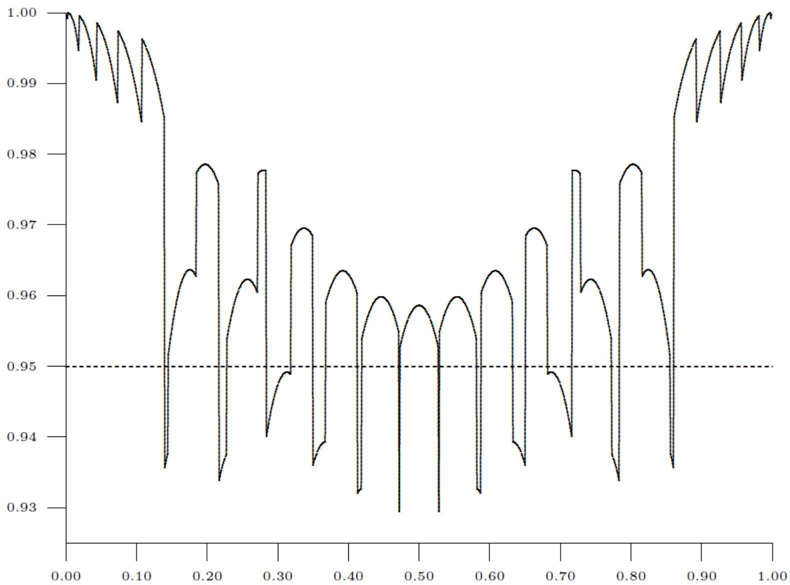

Figure 1: The confidence level of the shortest confidence interval: $n = 20$, $\gamma = 0.95$.

Note that for some probabilities θ the confidence level is smaller than the nominal one. This is in contradiction with the definition of the confidence interval (see Neyman 1934, Cramér 1949, Lehmann 1959, Silvey 1970).

In what follows, a small modification is introduced, after which the resulting shortest confidence interval does not have the above mentioned disadvantage, i.e. its confidence level is not smaller than the prescribed one.

Let Y be a random variable conditionally distributed on the interval $(0,1)$ with cdf $G_{Y|X=x}(\cdot)$. The confidence interval will be constructed on the basis of two random variables: $Z_s = X + Y$ and $Z_d = X - (1 - Y)$. The distributions of those r.v.'s are easy to obtain:

$$P_\theta\{Z_s \le t\} \;=\; \begin{cases} 0 & \text{if } t \le 0\,, \\ \alpha\big(\lfloor t\rfloor, \lceil t\rceil\big)\, P_\theta\{X = \lfloor t\rfloor\} & \text{if } \lfloor t\rfloor = 0\,, \\ \sum_{k=0}^{\lfloor t\rfloor-1} P_\theta\{X = k\} + \alpha\big(\lfloor t\rfloor, \lceil t\rceil\big)\, P_\theta\{X = \lfloor t\rfloor\} & \text{if } 1 \le \lfloor t\rfloor \le n\,, \\ 1 & \text{if } \lfloor t\rfloor > n\,, \end{cases}$$

$$P_\theta\{Z_d \le t\} = \begin{cases} 0 & \text{if } t \le -1\,, \\ \alpha\big(\lfloor t\rfloor, \lceil t\rceil\big)\, P_\theta\{X = \lfloor t\rfloor + 1\} & \text{if } \lfloor t\rfloor = -1\,, \\ \sum_{k=0}^{\lfloor t\rfloor} P_\theta\{X=k\} + \alpha\big(\lfloor t\rfloor + 1, \lceil t\rceil\big)\, P_\theta\{X = \lfloor t\rfloor + 1\} & \text{if } 0 \le \lfloor t\rfloor \le n-1\,, \\ 1 & \text{if } \lfloor t\rfloor \ge n\,, \end{cases}$$

where $\lfloor t\rfloor$ denotes the greatest integer no greater than t and

$$\lceil t\rceil = t - \lfloor t\rfloor \qquad \text{and} \qquad \alpha\big(\lfloor t\rfloor, \lceil t\rceil\big) = \int_0^{\lceil t\rceil} G_{Y|X=\lfloor t\rfloor}(du)\,.$$

It is easy to note that the distribution of Y may be taken as the uniform $U(0,1)$ independently of X.

The shortest confidence interval (θ_L, θ_U) at the confidence level γ will be obtained as a solution with respect to θ of the following problem:

$$\begin{cases} \theta_U - \theta_L = \min!\,, \\ P_{\theta_L}\big\{Z_s \le t\big\} = \gamma_2\,, \\ P_{\theta_U}\big\{Z_d \ge t\big\} = 1 - \gamma_1\,, \\ \gamma_2 - \gamma_1 = \gamma\,. \end{cases}$$

Hence, for observed $X = x$ and $Y = y$ we have to find θ_L and θ_U such that

$$\begin{cases} \theta_U - \theta_L = \min!\,, \\ \sum_{k=0}^{x-1} P_{\theta_L}\big\{X=k\big\} + y P_{\theta_L}\big\{X = x\big\} = \gamma_2\,, \\ \sum_{k=0}^{x} P_{\theta_U}\big\{X=k\big\} + y\, P_{\theta_U}\big\{X = x+1\big\} = \gamma_1\,, \\ \gamma_2 - \gamma_1 = \gamma\,, \end{cases}$$

or, equivalently,

$$\begin{cases} \theta_U - \theta_L = \min!\,, \\ (1-y)\, F\big(x, n-x+1; \theta_L\big) + y F\big(x+1, n-x; \theta_L\big) = \gamma_1\,, \\ (1-y)\, F\big(x+1, n-x; \theta_U\big) + y F\big(x+2, n-x-1; \theta_U\big) = \gamma + \gamma_1\,. \end{cases}$$

Let

$$G(\theta; n, x, y) = (1-y)\, F\big(x, n-x+1; \theta\big) + y F\big(x+1, n-x; \theta\big)\,.$$

We take $F(a, 0; \theta) = 0$ and $F(0, b; \theta) = 1$. Then

$$\theta_L = G^{-1}(\gamma_1; n, x, y) \qquad \text{and} \qquad \theta_U = G^{-1}(\gamma + \gamma_1; n, x+1, y)\,.$$

In what follows we consider only the case $x \le n/2$. If $x \ge n/2$, the role of success and failure should be interchanged.

The problem of finding the shortest confidence interval may be written as the problem of finding γ_1 which minimizes

$$d(\gamma_1; n, x, y) \;=\; G^{-1}(\gamma + \gamma_1; n, x+1, y) - G^{-1}(\gamma_1; n, x, y)$$

for given $y \in [0,1]$, n and x.

Theorem 2.1. *For $x \geq 2$ and for all $y \in (0,1)$ there exists a two-sided shortest confidence interval.*

Proof: We have to show that for $x \geq 2$ and for all $y \in (0,1)$ there exists $0 < \gamma_1 < 1-\gamma$ such that $d(\gamma_1; n, x, y)$ is minimal. The derivative of $d(\gamma_1; n, x, y)$ with respect to γ_1 equals

$$\frac{\partial d(\gamma_1; n, x, y)}{\partial \gamma_1} \;=\; \frac{1}{LHS(\gamma_1; n, x, y)} - \frac{1}{RHS(\gamma_1; n, x, y)}$$

where

$$\begin{aligned} LHS(\gamma_1; n, x, y) \;=\; & \Big(1 - G^{-1}\big(\gamma + \gamma_1; n, x+1, y\big)\Big)^{n-x-1} G^{-1}\big(\gamma + \gamma_1; n, x+1, y\big)^{x+1} \\ & \cdot \Bigg(\frac{1-y}{G^{-1}(\gamma + \gamma_1; n, x+1, y)\, B(x+1, n-x)} \\ & \quad + \; \frac{y}{\big(1 - G^{-1}(\gamma + \gamma_1; n, x+1, y)\big)\, B(x+2, n-x-1)} \Bigg) \end{aligned}$$

and

$$\begin{aligned} RHS(\gamma_1; n, x, y) \;=\; & \Big(1 - G^{-1}\big(\gamma_1; n, x, y\big)\Big)^{n-x} G^{-1}(\gamma_1; n, x, y)^{x} \\ & \cdot \Bigg(\frac{1-y}{G^{-1}(\gamma_1; n, x, y)\, B(x, n-x+1)} \\ & \quad + \; \frac{y}{\big(1 - G^{-1}(\gamma_1; n, x, y)\big)\, B(x+1, n-x)} \Bigg) . \end{aligned}$$

Because

$$G^{-1}(0; n, x, y) = 0 \qquad \text{and} \qquad G^{-1}(1; n, x, y) = 1 \,,$$

for $2 \leq x \leq n/2$ we have:

$$\begin{aligned} &\text{if } \; \gamma_1 \to 0 && \text{then } \; LHS(\gamma_1; n, x, y) > 0 && \text{and } \; RHS(\gamma_1; n, x, y) \to 0^+ \,, \\ &\text{if } \; \gamma_1 \to 1-\gamma && \text{then } \; LHS(\gamma_1; n, x, y) \to 0^+ && \text{and } \; RHS(\gamma_1; n, x, y) > 0 \,. \end{aligned}$$

Therefore, the equation

$$(*) \qquad \frac{\partial d(\gamma_1; n, x, y)}{\partial \gamma_1} \;=\; 0$$

has a solution. □

It is easy to see that $LHS(\cdot;n,x,y)$ and $RHS(\cdot;n,x,y)$ are unimodal and concave on the interval $(0,1-\gamma)$. Hence, the solution of $(*)$ is unique. Let γ_1^* denote the solution. Because $\frac{\partial d(\gamma_1;n,x,y)}{\partial\gamma_1} < 0$ for $\gamma_1 < \gamma_1^*$ and $\frac{\partial d(\gamma_1;n,x,y)}{\partial\gamma_1} > 0$ for $\gamma_1 > \gamma_1^*$, we have $d(\gamma_1^*;n,x,y) = \inf\bigl\{d(\gamma_1;n,x,y)\colon 0<\gamma_1<1-\gamma\bigr\}$.

Theorem 2.2. *For $x=1$ there exists $y^* \in (0,1)$ such that if $Y<y^*$ then the shortest confidence interval is one-sided, and it is two-sided otherwise.*

Proof: For $x=1$ we have

$$\frac{\partial d(\gamma_1;n,1,y)}{\partial\gamma_1} = \frac{1}{LHS(\gamma_1;n,1,y)} - \frac{1}{RHS(\gamma_1;n,1,y)}$$

where

$$\begin{aligned} LHS(\gamma_1;n,1,y) &= \Bigl(1-G^{-1}\bigl(\gamma+\gamma_1;n,2,y\bigr)\Bigr)^{n-2} G^{-1}\bigl(\gamma+\gamma_1;n,2,y\bigr)^2 \\ &\quad\cdot\Biggl(\frac{1-y}{G^{-1}(\gamma+\gamma_1;n,2,y)\,B(2,n-2)} \\ &\qquad + \frac{y}{\bigl(1-G^{-1}(\gamma+\gamma_1;n,2,y)\bigr)\,B(3,n-2)}\Biggr) \\ &= \frac{1}{2}\,(n-1)\,n\bigl(1-G^{-1}(\gamma+\gamma_1;n,2,y)\bigr)^{n-3} G^{-1}(\gamma+\gamma_1;n,2,y) \\ &\quad\cdot\Bigl(2\bigl(1-G^{-1}(\gamma+\gamma_1;n,2,y)\bigr) + y\bigl(n\,G^{-1}(\gamma+\gamma_1;n,2,y)-2\bigr)\Bigr) \end{aligned}$$

and

$$\begin{aligned} RHS(\gamma_1;n,1,y) &= \Bigl(1-G^{-1}(\gamma_1;n,1,y)\Bigr)^{n-1} G^{-1}(\gamma_1;n,1,y) \\ &\quad\cdot\Biggl(\frac{1-y}{G^{-1}(\gamma_1;n,1,y)\,B(1,n)} \\ &\qquad + \frac{y}{\bigl(1-G^{-1}(\gamma_1;n,1,y)\bigr)\,B(2,n-1)}\Biggr) \\ &= n\bigl(1-G^{-1}(\gamma_1;n,1,y)\bigr)^{n-2} \\ &\quad\cdot\Bigl(1-G^{-1}(\gamma_1;n,1,y) + y\bigl(n\,G^{-1}(\gamma_1;n,1,y)-1\bigr)\Bigr)\,. \end{aligned}$$

It can be seen that if $\gamma_1 \to 0$, then

$$\begin{aligned} LHS(\gamma_1;n,1,y) &\to \frac{\bigl(1-G^{-1}(\gamma;n,2,y)\bigr)^{n-3} G^{-1}(\gamma;n,2,y)}{B(2,n-1)} \\ &\quad\cdot\Bigl(2\bigl(1-G^{-1}(\gamma;n,2,y)\bigr) + y\bigl(n\,G^{-1}(\gamma;n,2,y)-2\bigr)\Bigr)\,, \\ RHS(\gamma_1;n,1,y) &\to (1-y)\,n\,. \end{aligned}$$

If $\gamma_1 \to 1-\gamma$, then

$$LHS(\gamma_1;n,1,y) \to 0^+\,,$$
$$RHS(\gamma_1;n,1,y) > 0\,.$$

Because $LHS(0;n,1,0) < RHS(0;n,1,0)$ and $LHS(0;n,1,1) > RHS(0;n,1,1)$, there exists y^* such that $LHS(0;n,1,y^*) = RHS(0;n,1,y^*)$. So, for $y < y^*$ the shortest confidence interval is one-sided, and it is two-sided otherwise. □

The value of y^* may be found numerically as a solution of

$$LHS(0;n,1,y^*) = RHS(0;n,1,y^*)\ .$$

In Table 2 the values of y^* for different n and confidence levels γ are given.

Table 2: Values of y^*.

n	$\gamma=0.9$	$\gamma=0.95$	$\gamma=0.99$	$\gamma=0.999$
10	0.783163	0.870995	0.964326	0.994792
20	0.828155	0.904080	0.976758	0.997138
30	0.840388	0.912599	0.979647	0.997620
40	0.846076	0.916491	0.980924	0.997825
50	0.849360	0.918718	0.981643	0.997937
60	0.851499	0.920160	0.982104	0.998009
70	0.853002	0.921170	0.982424	0.998058
80	0.854117	0.921917	0.982660	0.998094
90	0.854976	0.922491	0.982840	0.998121
100	0.855658	0.922947	0.982983	0.998143
150	0.857680	0.924294	0.983403	0.998206
200	0.858678	0.924955	0.983608	0.998237
300	0.859666	0.925610	0.983810	0.998267
400	0.860156	0.925934	0.983910	0.998281
500	0.860449	0.926128	0.983969	0.998290
600	0.860644	0.926257	0.984009	0.998296
700	0.860784	0.926349	0.984037	0.998300
800	0.860888	0.926418	0.984058	0.998303
900	0.860969	0.926471	0.984075	0.998306
1000	0.861034	0.926514	0.984088	0.998308

The above considerations may be summarized as follows. Observe a r.v. X distributed as $Bin(n,\theta)$ and draw Y distributed as $U(0,1)$. If $X > n/2$ then consider $X' = n-X$. Calculate y^*, the solution of the equation $LHS(y^*;0,n,1) = RHS(y^*;0,n,1)$.

If $X+Y \le 1+y^*$ then the confidence interval is of the form

$$\Big(0;\, G^{-1}(\gamma;n,X+1,Y)\Big)\ .$$

If $X+Y \ge 1+y^*$ then find γ_1^* which minimizes $d(\gamma_1;n,x,y)$. Then the confidence interval takes on the form

$$\Big(G^{-1}(\gamma_1^*;n,X,Y);\, G^{-1}(\gamma+\gamma_1^*;n,X+1,Y)\Big)\ .$$

If $X > n/2$ is observed then the shortest confidence interval has the form

$$\begin{cases} \left(1 - G^{-1}(\gamma; n, X'+1, Y); 1\right) & \text{if } X'+Y \le 1+y^*\,, \\ \left(1 - G^{-1}(\gamma+\gamma_1^*; n, X'+1, Y);\ 1 - G^{-1}(\gamma_1^*; n, X', Y)\right) & \text{otherwise}\,. \end{cases}$$

Theorem 2.3. *$P_\theta\{\theta_L \le \theta \le \theta_U\} \ge \gamma$ for $\theta \in (0,1)$, and $P_{0.5}\{\theta_L \le 0.5 \le \theta_U\} = \gamma$.*

Proof: Let $\theta \in (0,1)$ be given. Let x_u, y_u and γ_u be such that $\theta = G^{-1}(\gamma+\gamma_u; n, x_u, y_u)$. Similarly, let x_d, y_d and γ_d be such that $\theta = G^{-1}(\gamma_d; n, x_d+1, y_d)$. Of course, $x_d < x_u$ and $\gamma_d \le \gamma_u$. So

$$P_\theta\{\theta_L \le \theta \le \theta_U\} \;=\; P_\theta\{x_d + y_d \le X + Y \le x_u + y_u\} \;=\; \gamma + \gamma_u - \gamma_d \;\ge\; \gamma\,.$$

If $\theta = 0.5$ then, by symmetry, $x_u = n - x_d$, $y_u = 1 - y_d$ and $\gamma_u = \gamma_d$. Hence $P_{0.5}\{\theta_L \le 0.5 \le \theta_U\} = \gamma$. □

The confidence level of the randomized shortest confidence interval for $n = 20$ and $\gamma = 0.95$ is shown in Figure 2.

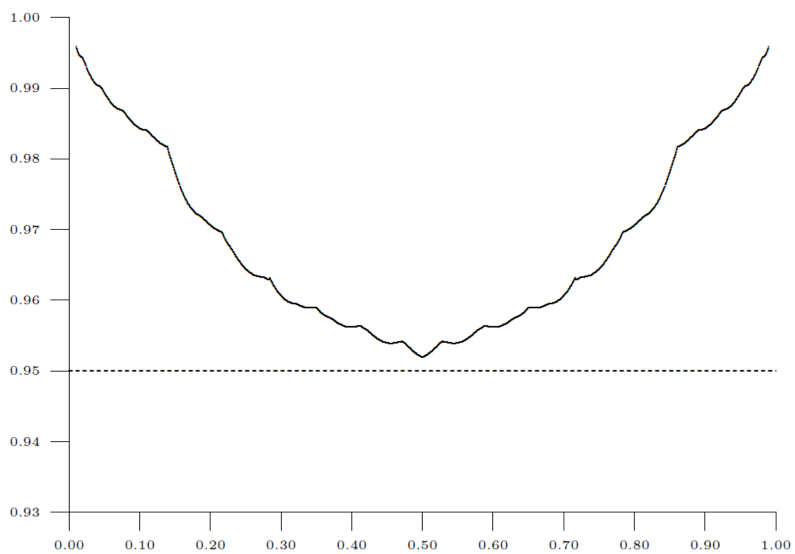

Figure 2: The confidence level of the randomized shortest confidence interval: $n = 20$, $\gamma = 0.95$.

In Clopper and Pearson's times, calculating quantiles of a beta distribution was numerically complicated. Nowadays, it is very easy with the aid of computer software, so using the shortest confidence interval is recommended (a short Mathematica program is given in the Appendix). To avoid problems with wrong inference due to the confidence level, one should use randomized shortest confidence intervals. Of course, the generated value y of a $U(0,1)$ r.v. must be attached to the final report. So results now are given by three numbers: number of trials, number of successes and the value y.

3. AN EXAMPLE

Consider an experiment consisting of $n = 20$ Bernoulli trials in which $x = 3$ successes were observed. Let $\gamma = 0.95$. The standard Clopper–Pearson confidence interval $\left(F^{-1}(3, 18; 0.025);\ F^{-1}(4, 17; 0.975)\right)$ takes on the form

$$(0.0321,\ 0.3789)\ .$$

The length of the standard Clopper–Pearson confidence interval equals 0.3468.

To calculate the randomized shortest confidence interval one has to draw a value y of the auxiliary variable Y and then calculate the ends of the confidence interval on the basis of $x + y$. The uniform random number generator gives $y = 0.0102162$ and the randomized shortest confidence interval takes on the form

$$(0.0184,\ 0.2898)\ .$$

The length of that confidence interval is 0.2714. Note that the length of the proposed confidence interval equals 78% of the length of the standard confidence interval.

The final report may look as follows:

$$n = 20\,,\quad x = 3\,,\quad y = 0.0102162\,,\quad \gamma = 0.95\,,\quad \theta \in (0.0184, 0.2898)\ .$$

In practical applications it is important to have conclusions as precise as possible. Hence the use of the randomized shortest confidence intervals is recommended, especially for small sample sizes. Those intervals are very easy to obtain with the aid of the standard computer software (see Appendix).

APPENDIX

Below we give a short Mathematica program for calculating γ_1^* and the ends of the randomized shortest confidence interval. Of course, one can also use other mathematical or statistical packages (in a similar way) to find the values of γ_1^*.

```
In[1]:= << Statistics'ContinuousDistributions'
     n=.; x=.; y=.; q=.;
     Bet[a_,b_,x_]=CDF[BetaDistribution[a, b], x];
     G[θ_,n_,x_,y_]=(1-y)*If[x==0,0,Bet[x,n-x+1,θ]]+y*If[x==n,0,Bet[x+1,n-x,θ]];
(*definition of the confidence interval*)
     Lower[p_,n_,x_,y_]:=If[x<=1+Ystar,0,θ/.FindRoot[G[θ,n,x,y]==p,{θ,0.001,0.999}];
     Upper[p_,n_,x_,y_]:=If[x>=n-1-Ystar,1,θ/.FindRoot[G[θ,n,x,y]==p,{θ,0.001,0.999}];
     Length[p_,n_,x_,y_,γ_]:=Upper[γ+p,n,x+1,y]-Lower[p,n,x,y];
In[2]:= n=20;(*input n*)
     x=7;(*input x (≤n/2)*)
     q=0.95 ;(*input confidence level*)
     y=RandomReal[];
(*calculate Y star*)
     eps=10^(-10); al=0; ar=1;
     While[ar-al>eps,{
      aa=(ar+al)/2;
      Dol=θ/.FindRoot[G[θ,n,2,aa]==q, {θ, 0.001, 0.999}];
      LHS=(1-Dol)^(n-3)*Dol*(2*(1-Dol)+aa*(n*Dol-2))/Beta[2,n-1];
      RHS=(1-aa)*n;
      If[LHS>RHS,ar=aa,al=aa];}]
     Ystar=aa;
(*calculate ends of the shortest confidence interval*)
     pp=If[x+y<=1+Ystar,0,
       p/.FindMinimum[Length[p,n,x,y,q],{p,0,1-q }][[2]]] (*probability γ1*)
     Left=Lower[pp,n,x,y] (*left end*)
     Right=Upper[q+pp,n,x,y] (*right end*)
     y (*drawn U(0,1) r.v.*)
```

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