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BIVARIATE BETA AND KUMARASWAMY MODELS DEVELOPED USING THE ARNOLD-NG BIVARIATE BETA DISTRIBUTION

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Abstract:

- In this paper we explore some mechanisms for constructing bivariate and multivariate beta and Kumaraswamy distributions. Specifically, we focus our attention on the Arnold-Ng (2011) eight parameter bivariate beta model. Several models in the literature are identified as special cases of this distribution including the Jones-Olkin-Liu-Libby-Novick bivariate beta model, and certain Kotz and Nadarajah bivariate beta models among others. The utility of such models in constructing bivariate Kumaraswamy models is investigated. Structural properties of such derived models are studied. Parameter estimation for the models is also discussed. For illustrative purposes, a real life data set is considered to exhibit the applicability of these models in comparison with rival bivariate beta and Kumaraswamy models.

Key-Words:

- *bivariate beta, second kind beta distribution, gamma components, copulas, bivariate Kumaraswamy models.*

AMS Subject Classification:

- 62E, 60F.

1. INTRODUCTION

Kumaraswamy (1980) introduced a two parameter absolutely continuous distribution which compares extremely favorably, in terms of simplicity, with the beta distribution. The Kumaraswamy distribution (hereafter the K distribution) on the interval $(0,1)$, has its probability density function (pdf) and its cdf with two shape parameters $\delta > 0$ and $\beta > 0$ defined by

$$f(x) = \delta\beta x^{\delta-1}(1-x^{\delta})^{\beta-1} I(0 < x < 1) \quad \text{and } F(x) = 1-(1-x^{\delta})^{\beta}. \tag{1.1}$$

If a random variable X has (1.1) as its density then we will write $X \sim K(\delta, \beta)$.

The density function in (1.1) has similar properties to those of the beta distribution. The Kumaraswamy pdf is unimodal, uniantimodal, increasing, decreasing or constant depending (similar to the beta distribution) on the values of the parameters. The construction of bivariate Kumaraswamy distributions has received limited attention.

Barreto-Souza and Lemonte (2013) introduced a bivariate Kumaraswamy distribution related to a Marshall-Olkin survival copula. They discussed some structural properties of their bivariate Kumaraswamy distribution, including a detailed discussion of estimation of the model parameters. Recently Arnold and Ghosh (2016) discussed some different strategies for constructing legitimate bivariate Kumaraswamy models via conditional specification, conditional survival function specification and via a copula based approach. In this paper, we consider several specialized approaches to the problem of constructing bivariate K distributions based on sub-models of the Arnold-Ng 8-parameter bivariate beta distribution. Included is discussion of the Jones-Olkin-Liu-Libby-Novick bivariate beta distribution and two Kotz and Nadarajah (2007) bivariate beta models.

To carry out this program, we make use of the observation that a Kumaraswamy distribution is a special case of the generalized beta distribution which is that of a positive power of a beta random variable.

In this paper we will make repeated use of the fact that a Kumaraswamy variable can be viewed as a power of a beta variable. Thus,

$$\text{if } \; Y \sim Beta(1, \beta), \quad \text{then for } \; \delta > 0, \quad X = Y^{1/\delta} \sim K(\delta, \beta).$$

Our proposed flexible families of bivariate Kumaraswamy distributions will be obtained by applying such marginal power transformations to suitable bivariate beta models. It will be convenient to begin with a careful discussion of the 8-parameter bivariate beta distribution introduced by Arnold and Ng (2011), together with its related sub-models and possible higher dimensional extensions. Note that in Arnold and Ghosh (2016), use was made of the simpler 5- parameter Arnold-Ng model in an analogous program for developing bivariate Kumaraswamy models. The present paper thus represents a natural extension of some of the results in that earlier paper.

We will begin with a detailed discussion of the 8-parameter Arnold-Ng model with marginals of the second kind beta type. The corresponding models with classical (first kind) beta marginals are then obtained via simple marginal transformations, using the observation that

$$\text{if } \; U \sim Beta^{(2)}(\alpha,\beta) \; \text{ then } \; (1+U^{-1})^{-1} \sim Beta(\alpha,\beta).$$

With suitable parametric restrictions, corresponding bivariate Kumaraswamy models are then readily derived.

The remainder of this article is organized as follows: In section 2, as mentioned above, we review the Arnold-Ng (2011) eight parameter bivariate second kind beta model. We consider the many sub-models that are obtained via parametric restrictions, and we discuss higher dimensional versions of this model. In section 3, we discuss the parallel models with marginals that are of the first or classical beta kind. In Section 4, we briefly consider the construction of bivariate generalized beta distributions. In Section 5, the useful concepts of reciprocation closure and closure under reflection about the point 1/2 are reviewed. Section 6 deals with a catalog of bivariate Kumaraswamy distributions obtained via marginal power transformations applied to certain bivariate beta variables. In section 7, we revisit the concept of reflection about 1/2. Section 8 includes some discussion of possible parameter estimation strategies for the models. Section 9 includes an illustrative application in which one of the bivariate Kumaraswamy models is compared with some competing models when fitted to a particular data set. Some concluding remarks are contained in Section 10.

2. BIVARIATE SECOND KIND BETA DISTRIBUTIONS

A random variable X is said to have a second kind beta distribution with positive parameters α_1 and α_2, if its density is of the form

$$f_X(x) = \frac{1}{B(\alpha_1,\alpha_2)} \frac{x^{\alpha_1-1}}{(1+x)^{\alpha_1+\alpha_2}} I(x>0)$$

and, in such a case, we write $X \sim B^{(2)}(\alpha_1,\alpha_2)$.

In our subsequent discussion we make considerable use of the observation that if V_1, V_2 are independent gamma distributed random variables with $V_i \sim \Gamma(\alpha_i, 1)$, $i = 1,2$, then $X = V_1/V_2 \sim B^{(2)}(\alpha_1,\alpha_2)$.

The construction of the Arnold-Ng (2011) 8-parameter bivariate second kind beta distribution begins with 8 independent gamma distributed random variables, $U_1, U_2, U_3, U_4, U_5, U_6, U_7, U_8$ with $U_i \sim \Gamma(\alpha_i, 1)$, $i = 1,2,...,8$. The random vector (V_1, V_2) is then defined by

$$V_1 = \frac{U_1+U_5+U_7}{U_3+U_6+U_8} \tag{2.1}$$

and

$$V_2 = \frac{U_2 + U_5 + U_8}{U_4 + U_6 + U_7}. \tag{2.2}$$

Utilizing the fact that sums of independent gamma variables with the same scale parameter again are gamma distributed, we see that

$$V_1 \sim B^{(2)}(\alpha_1 + \alpha_5 + \alpha_7, \alpha_3 + \alpha_6 + \alpha_8)$$

and

$$V_2 \sim B^{(2)}(\alpha_2 + \alpha_5 + \alpha_8, \alpha_4 + \alpha_6 + \alpha_7).$$

This 8-parameter model is the most general bivariate second kind beta model that can be constructed via ratios of sums of independent gamma variables. Some indication of how it was developed will be useful for envisioning how to construct higher dimensional versions. There are four places in which a gamma distributed variable can appear in (2.1)-(2.2). They are: in the numerator of (2.1), in the denominator of (2.1), in the numerator of (2.2) and in the denominator of (2.2). A random variable U might appear only once in the two ratios. This is the case for U_1, U_2, U_3 and U_4, each of which appears in a different one of the four available places. A random variable U might appear in two of the available four places, but to retain the independence of the numerator from its corresponding denominator in a given ratio, the same U cannot appear in both. There are four different ways in which a variable U can appear in two places as illustrated by U_5, U_6, U_7, and U_8. Thus U_5 appears in both numerators, U_6 appears in both denominators, etc.. A random variable U cannot appear in more two of the four places without violating the required independence of numerators from their corresponding denominators. This thus results in the appearance of the 8 U_i's in the general model (2.1)-(2.2). The addition of any more gamma distributed U's to the model in any one or two places will not yield a more general model since they could be combined with already present gamma variables to keep the dimension of the parameter vector at the value 8.

The three dimensional version of this construction will involve 26 U_i's. This number can be verified by noting that a trivariate model (V_1, V_2, V_3) expressed as ratios of independent linear combinations of independent gamma variables (with unit scale parameter), will involve 6 places where a particular U can appear, three numerators and three denominators. But a particular U cannot appear in both the numerator and denominator of any of the three V_i's. There will be 6 U's which appear in one of the 6 possible places. These will be denoted by $U_1, U_2, ..., U_6$. There will be 12 U's that appear in exactly two of the 6 possible positions, denoted by $U_7, U_8, ..., U_{18}$. Finally there are 8 U's that appear in 3 places, namely $U_{19}, U_{20}, ..., U_{26}$. No U can appear in more than 3 places without violating the requirement that numerators must be independent of their corresponding denominators.

Thus, there are a total of 26 parameters in the model where $U_i,\ i = 1, 2, ..., 26$ are independent variables with $U_i \sim \Gamma(\alpha_i, 1)$ for each i. The model can then be expressed in the following, somewhat daunting, form.

$$V_1 = \frac{U_1 + U_7 + U_8 + U_9 + U_{10} + U_{19} + U_{20} + U_{21} + U_{22}}{U_4 + U_{11} + U_{12} + U_{13} + U_{14} + U_{23} + U_{24} + U_{25} + U_{26}},$$

$$V_2 = \frac{U_2 + U_7 + U_{11} + U_{15} + U_{16} + U_{19} + U_{20} + U_{23} + U_{24}}{U_5 + U_9 + U_{13} + U_{17} + U_{18} + U_{21} + U_{22} + U_{25} + U_{26}},$$

and

$$V_3 = \frac{U_3 + U_8 + U_{12} + U_{15} + U_{17} + U_{19} + U_{21} + U_{23} + U_{25}}{U_6 + U_{10} + U_{14} + U_{16} + U_{18} + U_{20} + U_{22} + U_{24} + U_{26}}.$$

The pattern for the dimensions of parameter spaces of the multivariate models becomes clear. The univariate model involves 2 U's, i.e., $3^1 - 1$. The bivariate model involves 8 U's, i.e., $3^2 - 1$. The trivariate case involves 26 U's, i.e., $3^3 - 1$., and so on. The general four dimensional model has 80 parameters! The enormous number of parameters involved in the completely general 3 and 4 dimensional models (i.e., 26 and 80) will compel us to consider simplified sub-models, of somewhat restricted flexibility, obtained by setting some of the α's equal to 0. This may well be desirable, even in the bivariate case. The full array of sub-models of the 8 parameter model (2.1)-(2.2) can be enumerated as follows.

There is, to begin with, the full 8-parameter model in which all of the α_i's are positive. We can label the various sub-models by listing the subscripts of the α_i's which remain in the sub-model, i.e., which have not been set equal to 0. Thus $B^{(2)}(1, 2, 3, 4, 5.6, 7, 8)$ denotes the full model, while for example $B^{(2)}(1, 5, 6)$ denotes the model in which only α_1, α_5 and α_6 have not been set equal to 0. Note that the list of subscripts of the α_i's that are set equal to zero cannot include any of the four triples $(1, 5, 7), (3, 6, 8), (2, 5, 8)$ or $(4, 6, 7)$ in order to retain the second kind beta form for the marginal distributions. Thus, the list of permissible sub-models includes:

- $\binom{8}{1} = 8$ models in which just one of the α_i's has been set equal to 0,
- $\binom{8}{2} = 28$ models in which exactly two of the α_i's have been set equal to 0,
- $\binom{8}{3} - 4 = 52$ permissible models in which exactly three of the α_i's have been set equal to 0,
- $\binom{8}{4} - 20 = 50$ permissible models in which exactly four of the α_i's have been set equal to 0,
- $\binom{8}{5} - 36 = 20$ permissible models in which exactly five of the α_i's have been set equal to 0,
- $\binom{8}{6} - 26 = 2$ permissible models in which exactly six of the α_i's have been set equal to 0. Of these models, $BB^{(2)}(5, 6)$ has $V_1 = V_2$, while $BB^{(2)}(7, 8)$ has $V_1 = 1/V_2$, so that they are of little interest.

In all there are 161 models which might be considered, of which 159 are non trivial. As we shall see in the next section, several of the corresponding bivariate beta of the first kind models (but not many) have received detailed coverage in the literature. It should be noted that very few of these models have available analytic expressions for the corresponding joint density. Typically those models with more than 3 parameters will not have tractable joint densities.

Returning to the general 8-parameter model (2.1)-(2.2), we may readily write down the moments of the V_i's since they have second kind beta distributions. Thus, for any integer j less than $\alpha_3+\alpha_6+\alpha_8$, we have

$$\begin{aligned} E(V_1^j) &= E[(U_1+U_5+U_7)^j]E[(U_3+U_6+U_8)^{-j}] \\ &= \frac{\Gamma(\alpha_1+\alpha_5+\alpha_7+j)}{\Gamma(\alpha_1+\alpha_5+\alpha_7)}\frac{\Gamma(\alpha_3+\alpha_6+\alpha_8-j)}{\Gamma(\alpha_3+\alpha_6+\alpha_8)}, \end{aligned}$$

and similarly, for any integer $k<\alpha_4+\alpha_6+\alpha_7$,

$$E(V_2^k) = \frac{\Gamma(\alpha_2+\alpha_5+\alpha_8+k)}{\Gamma(\alpha_2+\alpha_5+\alpha_8)}\frac{\Gamma(\alpha_4+\alpha_6+\alpha_7-k)}{\Gamma(\alpha_4+\alpha_6+\alpha_7)}.$$

Expressions for the variances are then readily written down. However mixed moments are more difficult to deal with. For example, we have

$$E(V_1V_2) = E\left[\left(\frac{U_1+U_5+U_7}{U_3+U_6+U_8}\right)\left(\frac{U_2+U_5+U_8}{U_4+U_6+U_7}\right)\right]$$

which appears to be difficult to evaluate analytically, unless most of the α_i's are equal to 0. Thus, analytic expressions for the covariance between V_1 and V_2 will be usually unavailable. Nevertheless, the covariance and any mixed moments of the form $E(V_1^\ell V_2^m)$ can be readily approximated by repeated simulation of the U_i's, thanks to the strong law of large numbers.

3. BIVARIATE BETA DISTRIBUTIONS (OF THE FIRST, OR CLASSICAL, KIND)

If $U\sim B^{(2)}(\alpha_1,\alpha_2)$, i.e., if $U =^d W_1/W_2$ where the W_i's are independent with $W_i\sim\Gamma(\alpha_i,1),\ i=1,2$, then the random variable $V=(1-U^{-1})^{-1}$ has a (classical) beta distribution or beta distribution of the first kind, and we denote this by $V\sim B(\alpha_1,\alpha_2)$. Here, V can be represented in the form

$$V=\frac{W_1}{W_1+W_2}.$$

Application of such a transformation to the marginals of the model (2.1)-(2.2) yields a parallel 8-parameter bivariate (classical) beta distribution with the following structure

$$W_1 = \frac{U_1 + U_5 + U_7}{(U_1 + U_5 + U_7) + (U_3 + U_6 + U_8)} \tag{3.1}$$

and

$$W_2 = \frac{U_2 + U_5 + U_8}{(U_2 + U_5 + U_8) + (U_4 + U_6 + U_7)}, \tag{3.2}$$

where the U_i's are independent gamma distributed random variables with $U_i \sim \Gamma(\alpha_i, 1),\ i = 1, 2, ..., 8$. In this case we write

$$(W_1, W_2) \sim BB(1, 2, 3, 4, 5, 6, 7, 8),$$

indicating that all 8 of the U_i's are involved in the distribution. This is the 8-parameter bivariate beta distribution introduced in Section 6.1 of Arnold and Ng (2011). As was the case for the bivariate beta of the second kind distribution discussed in Section 2, it will often be of interest to consider sub-models in which some of the α_i's are set equal to zero, so that the corresponding U_i's do not appear in the expressions (2.1) and (2.2). Thus for example the model $BB(1, 2, 6, 7, 8)$ may be recognized as the 5-parameter bivariate beta model discussed extensively in Arnold and Ng (2011), while the simpler 3-parameter models $BB(1, 2, 6)$, $BB(3, 5, 6)$, $BB(4, 5, 6)$ and $BB(6, 7, 8)$ have also appeared in the literature, as has the 4-parameter model $BB(5, 6, 7, 8)$.

The $BB(6, 7, 8)$ model is recognizable as a Dirichlet distribution, the $BB(1, 2, 6)$ model is identifiable as the Libby-Novak (1982)-Jones (2002)-Olkin-Liu (2003) model, the $BB(3, 5, 6)$ and $BB(4, 5, 6)$ models are the same as the first two models discussed in Nadarajah and Kotz (2005), and the $BB(5, 6, 7, 8)$ has been discussed by Olkin and Trikalinos (2015). Finally we mention that the BB(1,2,3,4,5,6) model was introduced by Magnussen (2004). Of course, not all bivariate beta models can be viewed as sub-models of (3.1)-(3.2). For example the third model in Nadarajah and Kotz (2005) (which is defined in terms of three independent beta variables) is not of this form, nor are the various copula based models obtained by marginally transforming quite arbitrary bivariate distributions to obtain beta marginals. Moreover some bivariate beta models, such as for example the one in Nadarajah (2007) only have beta marginals in special sub-cases.

In this setting also, there are 161 models which might be considered, of which 159 are non trivial. It, once more, should be noted that very few of these models have available analytic expressions for the corresponding joint density. Typically those models with more than 3 parameters will not have tractable joint densities.

Returning to the general 8-parameter model (3.1)-(3.2), we may readily write down the moments of the W_i's since they have (classical) beta distributions. Thus, for example, the means and variances are given by

$$E(W_1) = \frac{\alpha_1 + \alpha_5 + \alpha_7}{\alpha_1 + \alpha_5 + \alpha_7 + \alpha_3 + \alpha_6 + \alpha_8},$$

$$E(W_2) = \frac{\alpha_2+\alpha_5+\alpha_8}{\alpha_2+\alpha_5+\alpha_8+\alpha_4+\alpha_6+\alpha_7},$$

$$var(W_1) = \frac{(\alpha_1+\alpha_5+\alpha_7)(\alpha_3+\alpha_6+\alpha_8)}{(\alpha_1+\alpha_5+\alpha_7+\alpha_3+\alpha_6+\alpha_8)^2(\alpha_1+\alpha_5+\alpha_7+\alpha_3+\alpha_6+\alpha_8+1)},$$

and

$$var(W_2) = \frac{(\alpha_2+\alpha_5+\alpha_8)(\alpha_4+\alpha_6+\alpha_7)}{(\alpha_2+\alpha_5+\alpha_8+\alpha_4+\alpha_6+\alpha_7)^2(\alpha_2+\alpha_5+\alpha_8+\alpha_4+\alpha_6+\alpha_7+1)}.$$

Although expressions for the variances are readily written down, mixed moments are more difficult to deal with. For example, we have

$$E(W_1W_2) = E\left[\left(\frac{U_1+U_5+U_7}{U_1+U_5+U_7+U_3+U_6+U_8}\right)\left(\frac{U_2+U_5+U_8}{U_2+U_5+U_8+U_4+U_6+U_7}\right)\right]$$

which will be difficult to evaluate analytically, unless most of the α_i's are equal to 0. Thus, analytic expressions for the covariance between W_1 and W_2 will be usually unavailable. As was the case for the second kind beta models, this covariance and any mixed moments of the form $E(W_1^\ell W_2^m)$ can be readily approximated by repeated simulation of the U_i's, using the strong law of large numbers.

4. RECIPROCATION AND REFLECTION ABOUT 1/2

If $X \sim B(\alpha_1, \alpha_2)$ then it follows readily that $1 - X \sim B(\alpha_2, \alpha_1)$. Similarly, if $X \sim B^{(2)}(\alpha_1, \alpha_2)$ then $1/X \sim B^{(2)}(\alpha_2, \alpha_1)$. In words, the family of beta distributions is closed under reflection about the point 1/2, and the family of second kind beta distributions is closed under reciprocation. If one of these transformations is applied to one of the coordinates of a bivariate beta random variable, a new bivariate beta random variable will be obtained, but with a modified dependence structure. Thus if (W_1, W_2) has a bivariate beta distribution with positive correlation, then $(W_1, 1 - W_2)$ will again have a bivariate beta distribution, but now it will have negative correlation (since $cov(W_1, 1 - W_2) = cov(W_1, -W_2) = -cov(W_1, W_2)$). Similarly, if (W_1, W_2) has a bivariate second kind beta distribution, then $(W_1, 1/W_2)$ will again have a bivariate second kind beta distribution, but typically with correlation opposite in sign to that of (W_1, W_2).

The $BB^{(2)}(\alpha_1, \alpha_2, \alpha_3, \alpha_4, \alpha_5, \alpha_6, \alpha_7, \alpha_8)$ family of distributions is closed under marginal reciprocation and, likewise, the $BB(\alpha_1, \alpha_2, \alpha_3, \alpha_4, \alpha_5, \alpha_6, \alpha_7, \alpha_8)$ family of distributions is closed under marginal reflection about 0. Specifically we have:

If $(W_1, W_2) \sim BB^{(2)}(\alpha_1, \alpha_2, \alpha_3, \alpha_4, \alpha_5, \alpha_6, \alpha_7, \alpha_8)$, then

- $(W_1, 1/W_2) \sim BB^{(2)}(\alpha_1, \alpha_4, \alpha_3, \alpha_2, \alpha_7, \alpha_8, \alpha_5, \alpha_6)$,

- $(1/W_1, W_2) \sim BB^{(2)}(\alpha_3, \alpha_2, \alpha_1, \alpha_4, \alpha_8, \alpha_7, \alpha_6, \alpha_5)$,

and

- $(1/W_1, 1/W_2) \sim BB^{(2)}(\alpha_3, \alpha_4, \alpha_1, \alpha_2, \alpha_6, \alpha_5, \alpha_8, \alpha_7)$.

In a parallel fashion, if $(W_1, W_2) \sim BB(\alpha_1, \alpha_2, \alpha_3, \alpha_4, \alpha_5, \alpha_6, \alpha_7, \alpha_8)$, then

- $(W_1, 1 - W_2) \sim BB(\alpha_1, \alpha_4, \alpha_3, \alpha_2, \alpha_7, \alpha_8, \alpha_5, \alpha_6)$,
- $(1 - W_1, W_2) \sim BB(\alpha_3, \alpha_2, \alpha_1, \alpha_4, \alpha_8, \alpha_7, \alpha_6, \alpha_5)$,

and

- $(1 - W_1, 1 - W_2) \sim BB(\alpha_3, \alpha_4, \alpha_1, \alpha_2, \alpha_6, \alpha_5, \alpha_8, \alpha_7)$.

See Singapurwalla et al. (2016) for further discussion of bivariate beta models related by marginal reflection about 1/2.

5. BIVARIATE GENERALIZED BETA MODELS

If $X \sim B(\alpha_1, \alpha_2)$ then for $\gamma > 0$, $W = X^{1/\gamma}$ is said to have a generalized beta distribution, written

$$W \sim GB(\alpha_1, \alpha_2, \gamma).$$

Similarly, if $X \sim B^{(2)}(\alpha_1, \alpha_2)$ then for $\gamma > 0$, $W = X^{1/\gamma}$ is said to have a generalized second kind beta distribution, written

$$W \sim GB^{(2)}(\alpha_1, \alpha_2, \gamma).$$

Analogous generalizations of our bivariate beta models are defined as follows.

If $(V_1, V_2) \sim BB(\alpha_1, \alpha_2, \alpha_3, \alpha_4, \alpha_5, \alpha_6, \alpha_7, \alpha_8)$ and if $W_1 = V_1^{1/\gamma_1}$ and $W_2 = V_1^{1/\gamma_2}$ then (W_1, W_2) has a bivariate generalized beta distribution and we write

$$(W_1, W_2) \sim GBB(\alpha_1, \alpha_2, \alpha_3, \alpha_4, \alpha_5, \alpha_6, \alpha_7, \alpha_8; \gamma_1, \gamma_2).$$

Analogously, if $(V_1, V_2) \sim BB^{(2)}(\alpha_1, \alpha_2, \alpha_3, \alpha_4, \alpha_5, \alpha_6, \alpha_7, \alpha_8)$ and if $W_1 = V_1^{1/\gamma_1}$ and $W_2 = V_1^{1/\gamma_2}$ then (W_1, W_2) has a bivariate generalized second kind beta distribution and we write

$$(W_1, W_2) \sim GBB^{(2)}(\alpha_1, \alpha_2, \alpha_3, \alpha_4, \alpha_5, \alpha_6, \alpha_7, \alpha_8; \gamma_1, \gamma_2).$$

Additional flexibility for the bivariate generalized second kind beta distribution can be achieved by introducing location, scale and rotation parameters. Thus for $\underline{\mu} \in (-\infty, \infty)^2$ and a 2×2 matrix A, we will define (using column vectors)

$$\underline{Z} = \underline{\mu} + A\underline{W}$$

where $\underline{W} \sim GBB^{(2)}(\alpha_1, \alpha_2, \alpha_3, \alpha_4, \alpha_5, \alpha_6, \alpha_7, \alpha_8; \gamma_1, \gamma_2)$.

6. BIVARIATE KUMARASWAMY MODELS

If $X \sim B(1, \beta)$ and $Y = X^{1/\gamma}$, then Y is said to have a Kumaraswamy (1980) distribution, and we write $Y \sim K(\gamma, \beta)$. This distribution is a special case of the generalized beta distribution, but it has one attractive feature. Unlike other generalized beta distributions, the Kumaraswamy distribution has a simple analytic expression available for its distribution function. Thus, if $Y \sim K(\gamma, \beta)$ then

$$F_Y(y) = 1 - (1 - y^\gamma)^\beta I(0 < y < 1).$$

As a consequence, the Kumaraswamy distribution has emerged as a serious competitor to the beta distribution for modeling data taking values in the unit interval. In Arnold and Ghosh (2016), several bivariate Kumaraswamy distributions were discussed in some detail. In this Section we will focus on bivariate Kumaraswamy distributions that can be constructed by marginal power transformations applied to the 8-parameter Arnold-Ng bivariate beta model (3.1)-(3.2), incorporating the needed parametric restrictions to ensure that the marginal distributions of the bivariate beta model have their first parameters equal to 1.

Thus we begin with (V_1, V_2) having the distribution of the form (3.1)-(3.2), but with the following constraints on the α parameters.

$$\alpha_1 + \alpha_5 + \alpha_7 = 1 \tag{6.1}$$

and

$$\alpha_2 + \alpha_5 + \alpha_8 = 1, \tag{6.2}$$

to ensure that $V_1 \sim B(1, \alpha_3 + \alpha_6 + \alpha_8)$ and $V_2 \sim B(1, \alpha_4 + \alpha_6 + \alpha_7)$.

We then define

$$(W_1, W_2) = (V_1^{1/\delta_1}, V_2^{1/\delta_2}),$$

for positive parameters δ_1 and δ_2, to obtain a bivariate Kumaraswamy model, and we write

$$(W_1, W_2) \sim BK(\alpha_1, \alpha_2, \alpha_3, \alpha_4, \alpha_5, \alpha_6, \alpha_7, \alpha_8; \delta_1, \delta_2).$$

This appears to be a 10-parameter model but, because of the two parametric restrictions (6.1)-(6.2), the parameter space is actually of dimension 8. The parameters of the model, $\alpha_3, \alpha_4, \alpha_5, \alpha_6, \alpha_7, \alpha_8; \delta_1$ and δ_2, are constrained as follows:

- $\delta_1, \delta_2 > 0$,
- $\alpha_3, \alpha_4, \alpha_6 \in [0, \infty)$,
- $\alpha_7, \alpha_8 \in [0, 1]$,
- $0 \le \alpha_5 \le \min\{1-\alpha_7, 1-\alpha_8\}$,

while $\alpha_1 = 1-\alpha_5-\alpha_7$ and $\alpha_2 = 1-\alpha_5-\alpha_8$.

As was the case for the bivariate beta and the bivariate second kind beta models discussed in Sections 2 and 3, simplified and more manageable sub-models can be identified by setting some of the α parameters equal to 0. Below we consider in some detail some of these simplified models.

6.1. The Dirichlet bivariate Kumaraswamy model

For this model, we set $\alpha_1 = \alpha_2 = \alpha_3 = \alpha_4 = \alpha_5 = 0$ and, in order to satisfy (6.1)-(6.2), we set $\alpha_7 = \alpha_8 = 1$, while $\alpha_6 \in (0, \infty)$. This results in a three parameter bivariate Kumaraswamy distribution of the form

$$W_1 = \left(\frac{U_7}{U_6+U_7+U_8}\right)^{1/\delta_1},$$

$$W_2 = \left(\frac{U_8}{U_6+U_7+U_8}\right)^{1/\delta_2},$$

where $\delta_1, \delta_2 > 0$ and U_7, U_8 are i.i.d. $\Gamma(1,1)$ variables, while $U_6 \sim \Gamma(\alpha_6, 1)$ is independent of U_7 and U_8.

Since there is only one α parameter remaining in the model, we may drop the subscript "6" and write

$$\underline{W} \sim \textit{Dirichlet-BK}(\alpha, \delta_1, \delta_2).$$

The corresponding joint density is of the form

$$f_{\underline{W}}(\underline{w}) = \alpha(\alpha+1)\delta_1\delta_2 w_1^{\delta_1-1} w_2^{\delta_2-1}(1-w_1^{\delta_1}-w_2^{\delta_2})^{\alpha-1}\ I(w_1, w_2 > 0, w_1^{\delta_1}+w_2^{\delta_2} < 1)$$

The marginal densities are, by construction, of the Kumaraswamy type. Thus

$$f_{W_1}(w_1) = (\alpha+1)\delta_1 w_1^{\delta_1-1}\left(1-w_1^{\delta_1}\right)^{\alpha} \quad I(0 < w_1 < 1).$$

and

$$f_{W_2}(w_2) = (\alpha+1)\delta_2 w_2^{\delta_2-1}\left(1-w_2^{\delta_2}\right)^{\alpha} \quad I(0 < w_2 < 1).$$

The corresponding conditional densities correspond to scaled Kumaraswamy distributions. Thus, the conditional density of W_2 given $W_1 = w_1$ will be

$$\begin{aligned} f_{W_2|W_1}(w_2|w_1) &= \alpha\delta_2 \frac{\left(1 - w_1^{\delta_1} - w_2^{\delta_2}\right)^{\alpha-1}}{\left(1 - w_1^{\delta_1}\right)^{\alpha}} \\ &= \frac{\alpha\delta_2}{\left(1 - w_1^{\delta_1}\right)} \left(1 - \frac{w_2^{\delta_2}}{\left(1 - w_1^{\delta_1}\right)}\right)^{\alpha-1} \qquad I(0 < w_2 < (1 - w_1^{\delta_1})^{1/\delta_2}). \end{aligned}$$

An analogous expression is available for the conditional density of W_1 given $W_2 = w_2$.

Using known results for the Beta and the Dirichlet distribution, we may verify that

$$E(W_1^{\gamma_1}) = \frac{\Gamma(1 + \gamma_1\delta_1^{-1})\Gamma(2 + \alpha)}{\Gamma(2 + \alpha + \gamma_1\delta_1^{-1})},$$
$$E(W_2^{\gamma_2}) = \frac{\Gamma(1 + \gamma_2\delta_2^{-1})\Gamma(2 + \alpha)}{\Gamma(2 + \alpha + \gamma_2\delta_2^{-1})},$$

and

$$E(W_1^{\gamma_1} W_2^{\gamma_2}) = \frac{\Gamma(2 + \alpha)\Gamma(1 + \gamma_1\delta_1^{-1})\Gamma(1 + \gamma_2\delta_2^{-1})}{\Gamma(\alpha + 2 + \gamma_1\delta_1^{-1} + \gamma_2\delta_2^{-1})},$$

from which one can obtain the covariance and correlation (which, for this model, are necessarily non-positive).

By differentiating $\log f_{\underline{W}}(\underline{w})$ it is possible to locate the mode of this joint density. It will be located at the point (w_1^*, w_2^*) where

$$w_1^* = \left\{ \frac{\delta_2(\alpha - 1)}{(\alpha\delta_2 - 1)(\alpha\delta_1 - 1) + (1 - \delta_2)} \right\}^{1/\delta_1}.$$

and

$$w_2^* = \left\{ \frac{\delta_1(\alpha - 1)}{(\alpha\delta_2 - 1)(\alpha\delta_1 - 1) + (1 - \delta_1)} \right\}^{1/\delta_2},$$

provided that this point is an interior point of the support set, i.e., provided that

$$w_1^*, w_2^* > 0, \text{ and } w_1^{*\delta_1} + w_2^{*\delta_2} < 1.$$

In other cases, the mode will occur on the boundary of the support set.

It must be remarked that the restrictive nature of the support of this bivariate Kumaraswamy model will limit its potential for applications.

6.2. The Libby-Novick-Jones-Olkin-Liu bivariate Kumaraswamy model

For this model, we set $\alpha_3 = \alpha_4 = \alpha_5 = \alpha_7 = \alpha_8 = 0$ and, in order to satisfy (6.1)-(6.2), we set $\alpha_1 = \alpha_2 = 1$, while $\alpha_6 \in (0, \infty)$. This results in a three parameter bivariate Kumaraswamy distribution of the form

$$W_1 = \left(\frac{U_1}{U_1 + U_6}\right)^{1/\delta_1}, \tag{6.3}$$

$$W_2 = \left(\frac{U_2}{U_2 + U_6}\right)^{1/\delta_2}, \tag{6.4}$$

where $\delta_1, \delta_2 > 0$ and U_1, U_2 are i.i.d. $\Gamma(1,1)$ variables, while $U_6 \sim \Gamma(\alpha_6, 1)$ is independent of U_1 and U_2.

Since there is only one α parameter remaining in the model, here too we may drop the subscript "6" and write

$$\underline{W} \sim LNJOL\text{-}BK(\alpha, \delta_1, \delta_2).$$

The corresponding joint density is of the form

$$f_{\underline{W}}(\underline{w}) = \alpha(\alpha+1)\delta_1\delta_2 w_1^{\delta_1 - 1} w_2^{\delta_2 - 1} \frac{(1 - w_1^{\delta_1})^{\alpha}(1 - w_2^{\delta_2})^{\alpha}}{(1 - w_1^{\delta_1} w_2^{\delta_2})^{\alpha+2}} \; I(0 < w_1, w_2 < 1). \tag{6.5}$$

Since the W_i's can be represented as powers of Beta random variables we can easily get the following expressions for their moments.

$$E(W_i^{\gamma_i}) = \frac{\alpha\Gamma(\frac{\gamma_i}{\delta_i} + 1)}{\Gamma(\frac{\gamma_i}{\delta_i} + \alpha + 1)}, \quad i = 1, 2.$$

A simple expression for $E(W_1 W_2)$ is not available, although it is possible to provide a series expansion for it, and hence for the covariance. As Olkin and Liu (2003) noted in the bivariate beta case (with the δ_i's equal to one) it is possible to verify a strong version of positive dependence for this model. For two points $(w_1, w_2), (w_1', w_2')$ (with $w_1 < w_1', w_2 < w_2'$) it is readily verified that

$$\frac{f_{W_1,W_2}(w_1, w_2) f_{W_1,W_2}(w_1', w_2')}{f_{W_1,W_2}(w_1, w_2') f_{W_1,W_2}(w_1', w_2)} \geq 1,$$

so the joint density is positive likelihood ratio dependent. Consequently the correlation is always positive in this model.

6.3. The Nadarajah-Kotz bivariate Kumaraswamy model of the first kind

For this model, we set $\alpha_1 = \alpha_2 = \alpha_4 = \alpha_7 = \alpha_8 = 0$ and, in order to satisfy (6.1)-(6.2), we set $\alpha_5 = 1$, while $\alpha_3, \alpha_6 \in (0, \infty)$. This results in a four parameter bivariate Kumaraswamy distribution of the form

$$W_1 = \left(\frac{U_5}{U_3+U_5+U_6}\right)^{1/\delta_1},$$

$$W_2 = \left(\frac{U_5}{U_5+U_6}\right)^{1/\delta_2},$$

where $\delta_1, \delta_2 > 0$ and the U_i's are independent gamma variables with $U_5 \sim \Gamma(1,1)$, $U_3 \sim \Gamma(\alpha_3, 1)$ and $U_6 \sim \Gamma(\alpha_6, 1)$. In this case we write

$$\underline{W} \sim NK(1)\text{-}BK(\alpha_3, \alpha_6, \delta_1, \delta_2).$$

The corresponding joint density is of the form

$$f_{\underline{W}}(\underline{w}) = \alpha_6\delta_1\delta_2 \frac{(w_2^{\delta_2} - w_1^{\delta_1})^{\alpha_3-1}(1-w_2^{\delta_2})^{\alpha_6-1}}{w_1^{1-\delta_1} w_2^{\delta_2(\alpha_3+\alpha_6-1)+1} B(\alpha_6+1, \alpha_3)} \; I(0 < w_1^{\delta_1} < w_2^{\delta_2} < 1)\,.$$

Because of the structure of the $NK(1)$ bivariate beta model, it is possible to obtain expressions for arbitrary mixed moments as follows. For arbitrary $\tau_1, \tau_2 > 0$, we have

$$\begin{aligned} E(W_1^{\tau_1}W_2^{\tau_2}) &= E\left(\left(\frac{U_5}{U_3+U_5+U_6}\right)^{\tau_1/\delta_1}\left(\frac{U_5}{U_5+U_6}\right)^{\tau_2/\delta_2}\right) \\ &= E\left(\left(\frac{U_5}{U_5+U_6}\frac{U_5+U_6}{U_3+U_5+U_6}\right)^{\tau_1/\delta_1}\left(\frac{U_5}{U_5+U_6}\right)^{\tau_2/\delta_2}\right), \end{aligned}$$

where $U_5/(U_5{+}U_6)$ and $(U_5{+}U_6)/(U_3{+}U_5{+}U_6)$ are independent beta distributed random variables. Thus

$$\begin{aligned} E(W_1^{\gamma_1}W_2^{\gamma_2}) &= E\left(\left(\frac{U_5}{U_5+U_6}\right)^{(\gamma_1/\delta_1)+(\gamma_2/\delta_2)}\right) E\left(\left(\frac{U_5+U_6}{U_3+U_5+U_6}\right)^{\gamma_1/\delta_1}\right) \\ &= \frac{B(1+(\gamma_1/\delta_1)+(\gamma_2/\delta_2), \alpha_6)}{B(1,\alpha_6)} \frac{B(1+\alpha_6+(\gamma_1/\delta_1), \alpha_3)}{B(1+\alpha_6,\alpha_3)}. \end{aligned}$$

From this we may obtain the following expression for the covariance in this model

$$\begin{aligned} Cov(W_1, W_2) &= E(W_1W_2) - E(W_1)E(W_2) \\ &= \left(\frac{B(1+1/\delta_1+1/\delta_2, \alpha_6)}{B(1,\alpha_6)}\right)\left(\frac{B(1+1/\delta_1+\alpha_6, \alpha_3)}{B(1+\alpha_6,\alpha_3)}\right) \\ &\quad - \left(\frac{B(1+1/\delta_1, \alpha_3+\alpha_6)}{B(1,\alpha_3+\alpha_6)}\right)\left(\frac{B(1+1/\delta_2, \alpha_6)}{B(1,\alpha_6)}\right). \end{aligned}$$

In the special case in which $\delta_1 = \delta_2 = 1$, it is possible to verify that this covariance is always non-negative. For other values of the δ's, negative covariance is possible. Sufficient conditions for negative covariance (and hence, correlation) are that

$$\frac{1}{\delta_2} > \max(\alpha_6, \delta_1), \quad \alpha_6 > \alpha_3 \quad \text{and} \quad \alpha_3 + \alpha_6 > \frac{1}{\delta_1} > (\alpha_6 - 1).$$

By differentiating $\log f_{\underline{W}}(\underline{w})$ it is possible to locate the mode of this joint density. It will be located at the point (w_1^*, w_2^*) where

$$w_1^{*\delta_1} = \frac{w_2^{*\delta_2}(\delta_1 - 1)}{(\alpha_3\delta_1 - 1)}$$

and

$$w_2^* = \left\{ \frac{(-1-\alpha_6\delta_2) + \frac{(-1+\delta_1)}{(\alpha_3\delta_1-1)}(1+(-1+\alpha_3+\alpha_6)\delta_2)}{(1+\delta_2) - \frac{(1+\alpha_3\delta_2)(-1+\delta_1)}{(\alpha_3\delta_1-1)}} \right\}^{1/\delta_2},$$

provided that this point is an interior point of the support set, i.e., provided that

$$0 < w_1^{*\delta_1} < w_2^{*\delta_2} < 1.$$

In other cases, the mode will occur on the boundary of the support set.

In this case too, unless $\delta_1 = \delta_2$, the restrictive nature of the support of this bivariate Kumaraswamy model will limit its potential for applications.

6.4. The Nadarajah-Kotz bivariate Kumaraswamy model of the second kind

For this model, we set $\alpha_1 = \alpha_2 = \alpha_3 = \alpha_7 = \alpha_8 = 0$ and, in order to satisfy (6.1)-(6.2), we set $\alpha_5 = 1$, while $\alpha_3, \alpha_6 \in (0, \infty)$. This results in a four parameter bivariate Kumaraswamy distribution of the form

$$W_1 = \left(\frac{U_5}{U_5 + U_6}\right)^{1/\delta_1},$$

$$W_2 = \left(\frac{U_5}{U_4 + U_5 + U_6}\right)^{1/\delta_2},$$

where $\delta_1, \delta_2 > 0$ and the U_i's are independent gamma variables with $U_5 \sim \Gamma(1,1)$, $U_4 \sim \Gamma(\alpha_4, 1)$ and $U_6 \sim \Gamma(\alpha_6, 1)$. However, this can be recognized as a re-parameterized version of the $NK(1)BK$ distribution, with the subscripts of the W_i's interchanged. It is thus not necessary to list expressions for the joint density, moments, etc., since that material can easily be gleaned from Section 6.3.

6.5. The Olkin-Trikalinos bivariate Kumaraswamy model

For this model, we set $\alpha_1 = \alpha_2 = \alpha_3 = \alpha_4 = 0$ and, in order to satisfy (6.1)-(6.2), we set $\alpha_5 \in (0, 1)$, while $\alpha_7 = \alpha_8 = 1 - \alpha_5$ and $\alpha_6 \in (0, \infty)$. This results in a four parameter bivariate Kumaraswamy distribution of the form

$$W_1 = \left(\frac{U_5 + U_7}{U_5 + U_6 + U_7 + U_8}\right)^{1/\delta_1},$$

$$W_2 = \left(\frac{U_5 + U_8}{U_5 + U_6 + U_7 + U_8} \right)^{1/\delta_2},$$

where $\delta_1, \delta_2 > 0$ and the U_i's are independent gamma variables with $U_i \sim \Gamma(\alpha_i, 1), \quad i = 5, 6, 7, 8$. In this case we write

$$\underline{W} \sim OT\text{-}BK(\alpha_5, \alpha_6, \delta_1, \delta_2).$$

In this case also, an analytic expression for the joint density is not available, but we can make the following observations about this joint distribution.

Marginal moments are of course Kumaraswamy moments and thus are readily written down. Mixed moments are more troublesome, except in the case when $\delta_1 = \delta_2 = 1$ in which case we reduce to an Olkin-Trikalinos model and the W_i's can be represented as sums of coordinates of a three dimensional Dirichlet variable. For example, in this case as observed by Olkin and Trikalinos, a simple expression for the covariance can be obtained in the following form

$$cov(W_1, W_2) = \frac{(\alpha_5\alpha_6 - \alpha_7\alpha_8)}{(\alpha_5 + \alpha_6 + \alpha_7 + \alpha_8)(\alpha_5 + \alpha_6 + \alpha_7 + \alpha_8 + 1)}, \tag{6.6}$$

when $\delta_1 = \delta_2 = 1$,

Recall that our model is:

$$(W_1, W_2) = \left(\left(\frac{U_5 + U_7}{U_5 + U_6 + U_7 + U_8} \right)^{1/\delta_1}, \left(\frac{U_5 + U_8}{U_5 + U_6 + U_7 + U_8} \right)^{1/\delta_2} \right)$$

where the U_j's are independent with

$$U_5 \sim \Gamma(\alpha_5, 1)\ , \alpha_5 \in (0, 1),\ U_6 \sim \Gamma(\alpha_6, 1)\ , \alpha_6 \in (0, \infty)$$

and

$$U_7 \sim \Gamma(1 - \alpha_5, 1),\ U_8 \sim \Gamma(1 - \alpha_5, 1).$$

To study moments of this distribution, consider the following three dimensional Dirichlet model, which has four positive parameters:

$$(Y_1, Y_2, Y_3) = \left(\frac{U_5}{U_5 + U_6 + U_7 + U_8}, \frac{U_7}{U_5 + U_6 + U_7 + U_8}, \frac{U_8}{U_5 + U_6 + U_7 + U_8} \right)$$

with a $Dirichlet(\alpha_5, 1 - \alpha_5, 1 - \alpha_5, \alpha_6)$ distribution. So we have available expressions for

$$E(Y_1), E(Y_2), E(Y_3), E(Y_1^2), E(Y_2^2), E(Y_3^2), E(Y_1 Y_2), E(Y_1 Y_3), E(Y_2 Y_3)$$

and indeed for

$$E(Y_1^{\tau_1}), E(Y_2^{\tau_2}), E(Y_3^{\tau_3}), E(Y_1^{\tau_1} Y_2^{\tau_2}), E(Y_1^{\tau_1} Y_3^{\tau_3}), E(Y_2^{\tau_2} Y_3^{\tau_3})$$

and for

$$E(Y_1^{\tau_1} Y_2^{\tau_2} Y_1 Y_3^{\tau_3}).$$

But note that

$$(W_1, W_2) = \left((Y_1 + Y_2)^{1/\delta_1}, (Y_1 + Y_3)^{1/\delta_2} \right).$$

In general only a series expansion for $E(W_1^{\nu_1} W_2^{\nu_2})$ will be available. However, in the unlikely case in which $\nu_1/\delta_1 = k_1$, a positive integer and $\nu_2/\delta_2 = k_2$ is also a positive integer then we can write:

$$\begin{aligned} E(W_1^{\nu_1} W_2^{\nu_2}) &= E\left((Y_1 + Y_2)^{\nu_1/\delta_1} (Y_1 + Y_3)^{\nu_2/\delta_2} \right) \\ &= E\left((Y_1 + Y_2)^{k_1} (Y_1 + Y_3)^{k_2} \right) \\ &= \sum_{\ell_1=0}^{k_1} \sum_{\ell_2=0}^{k_2} \binom{k_1}{\ell_1} \binom{k_2}{\ell_2} E(Y_1^{\ell_1+\ell_2} Y_2^{k_1-\ell_1} Y_3^{k_2-\ell_2}), \end{aligned}$$

which is then computable. In particular, if $\delta_1 = \delta_2 = 1$, we get

$$E(W_1 W_2) = E[(Y_1 + Y_2)(Y_1 + Y_3)] = E(Y_1^2) + E(Y_1 Y_2) + E(Y_1 Y_3) + E(Y_2 Y_3)$$

which is easy to evaluate and then subtracting $E(W_1)E(W_2)$ we eventually re-confirm the result in (6.6).

$$\begin{aligned} cov(W_1, W_2) &= \frac{(\alpha_5 \alpha_6 - \alpha_7 \alpha_8)}{(\alpha_5 + \alpha_6 + \alpha_7 + \alpha_8)(\alpha_5 + \alpha_6 + \alpha_7 + \alpha_8 + 1)} \\ &= \frac{[\alpha_5 \alpha_6 - (1 - \alpha_5)^2]}{(\alpha_6 - \alpha_5 + 2)(\alpha_6 - \alpha_5 + 3)} \end{aligned}$$

where we have imposed the constraints $\alpha_7 = \alpha_8 = 1 - \alpha_5$.

When $\delta_1 = \delta_2 = 1$, the model encompasses a full range of values for its covariance and correlation. In particular we have

- The OT-BK model with $\delta_1 = \delta_2 = 1$, will exhibit positive correlation if $\alpha_6 \geq \alpha_5 + 2$, and $\alpha_5 > 1/4$.
- The OT-BK model with $\delta_1 = \delta_2 = 1$, will exhibit negative correlation if $\alpha_6 \leq \alpha_5 - 3$, and $\alpha_5 < 1/4$.

More specifically, with $\delta_1 = \delta_2 = 1$,

- When $\alpha_5 = 0$, $Cov(W_1, W_2) = -\frac{1}{(\alpha_6+2)(\alpha_6+3)} < 0$, for any choice of $\alpha_6 \in (0, \infty)$.
- When $\alpha_5 = 1$, $Cov(W_1, W_2) = \frac{\alpha_6}{(\alpha_6+1)(\alpha_6+2)} > 0$, for any choice of $\alpha_6 \in (0, \infty)$.

In cases in which the δ's are not both equal to 1, the covariances and correlations will have to be evaluated numerically in order to determine when they are positive and when negative.

6.6. The Ghosh bivariate Kumaraswamy model

For this model, suggested by I. Ghosh, we set $\alpha_5 = \alpha_7 = \alpha_8 = 0$ and, in order to satisfy (6.1)-(6.2), we set $\alpha_1 = \alpha_2 = 1$, while $\alpha_3, \alpha_4, \alpha_6 \in (0, \infty)$. This results in a five parameter bivariate Kumaraswamy distribution of the form

$$W_1 = \left(\frac{U_1}{U_1 + U_3 + U_6}\right)^{1/\delta_1},$$

$$W_2 = \left(\frac{U_2}{U_2 + U_4 + U_6}\right)^{1/\delta_2},$$

where $\delta_1, \delta_2 > 0$ and the U_i's are independent gamma variables with $U_1, U_2 \sim \Gamma(1, 1)$ and $U_i \sim \Gamma(\alpha_i, 1), \quad i = 3, 4, 6$. In this case we write

$$\underline{W} \sim G\text{-}BK(\alpha_3, \alpha_4, \alpha_6, \delta_1, \delta_2).$$

In this case also, an analytic expression for the joint density is not available.

6.7. The Magnussen Kumaraswamy model

Magnussen (2004) described a bivariate beta distribution which can be identified as a special case of the Arnold-Ng(8) bivariate beta model, obtained by setting $\alpha_7 = \alpha_8 = 0$. It is thus of the form:

$$\left(\frac{U_1 + U_5}{U_1 + U_3 + U_5 + U_6}, \frac{U_2 + U_5}{U_2 + U_4 + U_5 + U_6}\right).$$

In order to satisfy (6.1)-(6.2), we must have $\alpha_1 + \alpha_5 = 1$ and $\alpha_2 + \alpha_5 = 1$, while $\alpha_3, \alpha_4, \alpha_6 \in (0, \infty)$. This results in a six parameter bivariate Kumaraswamy distribution of the form

$$W_1 = \left(\frac{U_1 + U_5}{U_1 + U_3 + U_5 + U_6}\right)^{1/\delta_1},$$

$$W_2 = \left(\frac{U_2 + U_5}{U_2 + U_4 + U_5 + U_6}\right)^{1/\delta_2},$$

where $\delta_1, \delta_2 > 0$ and the U_i's are independent gamma variables with $U_5 \sim \Gamma(\alpha_5, 1)$ where $\alpha_5 \in [0, 1]$, $U_i \sim \Gamma(1 - \alpha_5, 1), \quad i = 1, 2$ and $U_i \sim \Gamma(\alpha_i, 1), \alpha_i \in (0, \infty), \quad i = 3, 4, 6$. In this case we write

$$\underline{W} \sim M\text{-}BK(\alpha_3, \alpha_4, \alpha_5, \alpha_6, \delta_1, \delta_2).$$

7. VARIATIONS, USING REFLECTION ABOUT 1/2

It is possible to construct other bivariate Kumaraswamy models by applying one or two marginal reflections about the point 1/2 to the bivariate beta model, before imposing the necessary parameter constraints and the marginal power transformations. For example the model (6.3)-(6.4), was derived by first considering a bivariate beta model of the form

$$(V_1, V_2) = \left(\frac{U_1}{U_1 + U_6}, \frac{U_2}{U_2 + U_6} \right).$$

Instead, we can consider starting with the doubly reflected model, $(1-V_1, 1-V_2)$, i.e.,

$$\left(\frac{U_6}{U_1 + U_6}, \frac{U_6}{U_2 + U_6} \right).$$

However, note that, according to our notation of Section 4, U_1 is playing the role of a gamma variable usually denoted by U_3, U_2 is playing the role of a variable usually denoted by U_4, and U_6 would be better labeled U_5. Thus we eventually arrive at the following four parameter bivariate Kumaraswamy model

$$W_1 = \left(\frac{U_5}{U_3 + U_5} \right)^{1/\delta_1}, \quad W_2 = \left(\frac{U_5}{U_4 + U_5} \right)^{1/\delta_2},$$

where $\delta_1, \delta_2 > 0$ and the U_i's are independent gamma variables with $U_5 \sim \Gamma(1,1)$ and $U_i \sim \Gamma(\alpha_i, 1),\ \ i = 3, 4$. If, instead we only reflect V_2 about 1/2, we eventually arrive at the model,

$$W_1 = \left(\frac{U_1}{U_1 + U_8} \right)^{1/\delta_1}, \quad W_2 = \left(\frac{U_8}{U_4 + U_8} \right)^{1/\delta_2},$$

where $\delta_1, \delta_2 > 0$ and the U_i's are independent gamma variables with $U_i \sim \Gamma(1,1)\ \ i = 1, 8$ and $U_4 \sim \Gamma(\alpha_4, 1)$.

Finally, if we only reflect V_1 about 1/2, we eventually arrive at the model,

$$W_1 = \left(\frac{U_7}{U_3 + U_7} \right)^{1/\delta_1}, \quad W_2 = \left(\frac{U_2}{U_2 + U_7} \right)^{1/\delta_2},$$

where $\delta_1, \delta_2 > 0$ and the U_i's are independent gamma variables with $U_i \sim \Gamma(1,1),\ \ i = 2, 7$ and $U_3 \sim \Gamma(\alpha_3, 1)$.

This approach can be applied to each of the bivariate models discussed in this section to obtain three related but distinct models in each case. Recall that such modifications of the models may be useful since reflection of one of the coordinates in the model about 1/2 will typically change the sign of the correlations in the original model.

8. PARAMETER ESTIMATION

The reader will have noticed that many of the models discussed in this paper do not have available analytic expressions for their joint densities. In addition, in many cases, it is difficult to identify functions of (W_1, W_2), say $g(W_1, W_2)$ for which $E(g(W_1, W_2))$ can be evaluated as a tractable function of the parameters of the model. We do have well behaved marginal distributions with available densities and moments, since the coordinate variables have Beta, second kind Beta, generalized Beta or Kumaraswamy distributions. Exceptions to this rule are the Libby-Novick-Jones-Olkin-Liu models for which, at least, the joint density is available, though mixed moments are only available in series form. Having observed this, we recognize that the old standby's maximum likelihood and the method of moments will require some modification if they are to be used to provide estimates of the model parameters. The same can be said for Bayesian estimation since it, also, typically utilizes a likelihood function. Arnold and Ng (2011) described a hybrid estimation strategy for parameter estimation in a 5-parameter sub-model of the BB(1,2,3,4,5,6,7,8) model, namely the BB(1,2,6,7,8) model. Unfortunately, their approach will not work for the associated bivariate Kumaraswamy model. In addition, an approximate Bayesian analysis of the BB(1,2,6,7,8) model was presented in Crackel (2015).

However, all is not lost because, without exception, all of the models discussed in this paper are easy to simulate. This means that, for given values of the parameters, highly accurate approximate values of moments, mixed moments, values of the joint distribution function and values of the joint moment generating function can be obtained. Admittedly, this will result in computer intensive estimation strategies, but it will allow selection among the sub-models for the one best adapted to a given data set. More details on these approximate estimation strategies will be the subject of a subsequent report.

9. A DATA SET

To illustrate the applicability of the bivariate beta and Kumaraswamy models developed in this paper, we consider the following data from the official website of the United Nations Development Program which can be found at (datalink: http://hdr.undp.org/en/composite/trends.) It consists of data on the Human Development Index (HDI)and is provided by the United Nations Development Program (UNDP). Specifically, we look at the 49 countries which are labeled as having very high HDI values for two specific years, the years 2010 and 2014. The reason of choosing these two particular time periods is that 2010 is right after the global financial turmoil (which started during the year 2008) which affected the entire economic sphere and related development and 2014 is the period where most of the countries in Europe were getting out of a recession.

Thus, it is quite interesting to see the change in the HDI values among countries over this period of 4 years.

We consider the following: Let X denote the HDI value for these 49 countries for the year 2010 and Y be the same for the year 2014. Noticeably, all the data points are within the range $(0,1)$, thereby a reasonable approach will be to fit bivariate distributions on the unit square, $[0,1]^2$. At this point we argue that (X,Y) can be modeled well by the bivariate Kumaraswamy and beta distributions developed and discussed in this paper.

1. Model I: The Libby- Novick-Jones-Olkin-Liu bivariate Kumaraswamy distribution. This absolutely continuous distribution has the following density (repeating (6.5)

$$f_{\underline{W}}(\underline{w}) = \alpha(\alpha+1)\delta_1\delta_2 w_1^{\delta_1-1} w_2^{\delta_2-1} \frac{(1-w_1^{\delta_1})^{\alpha}(1-w_2^{\delta_2})^{\alpha}}{(1-w_1^{\delta_1}w_2^{\delta_2})^{\alpha+2}}\, I(0<w_1,w_2<1).$$

2. Model II: The bivariate generalized beta distribution of the first kind [Equation (20) of Sarabia et al. (2014)], with density

$$f(x,y) = \frac{a_1 a_2}{B(p_1,p_2,q)} \frac{x^{a_1p_1-1} y^{a_2p_2-1}\,(1-x^{a_1})^{p_2+q-1}\,(1-y^{a_2})^{p_1+q-1}}{(1-x^{a_1}y^{a_2})^{-(p_1+p_2+q)}},$$

for $0<(x,y)<1$, where $B(p_1,p_2,q)$ is the normalizing constant.

3. Model III: The Nadarajah (2007) bivariate generalized beta distribution given by

$$f(x,y) = \frac{Cx^{\alpha-1}y^{\beta-1}\,(1-x)^{\gamma-\alpha-1}\,(1-y)^{\gamma-\beta-1}}{(1-xy\delta)^{\gamma}},$$

for $0<x<1$, $0<y<1$, $\gamma>\alpha>0$, $\gamma>\beta>0$ and $0\leq\delta<1$ where C is the normalizing constant given by

$$\frac{1}{C} = \frac{\Gamma(\alpha)\Gamma(\beta)\Gamma(\gamma-\alpha)\Gamma(\gamma-\beta)}{\Gamma^2(\gamma)}\, {}_2F_1\left(\alpha,\beta;\gamma;\delta\right).$$

4. Model IV: Olkin & Liu (2003) bivariate beta distribution given by

$$f(x,y) = \frac{1}{B(\alpha_0,\alpha_1,\alpha_2)} \frac{x^{\alpha_1-1}y^{\alpha_2-1}\,(1-x)^{\alpha_0+\alpha_2-1}\,(1-y)^{\alpha_0+\alpha_1-1}}{(1-xy)^{\alpha_0+\alpha_1+\alpha_2}},$$

for $0<(x,y)<1$, where $B(\alpha_0,\alpha_1,\alpha_2) = \frac{\Gamma(\alpha_0)\Gamma(\alpha_1)\Gamma(\alpha_2)}{\Gamma(\alpha_0+\alpha_1+\alpha_2)}$.

The other bivariate beta Kumaraswamy models, are not considered in this application because either they do not have a closed form expression for the density, or if they have one, their support set does not match the range of points in the data set.

- The NK- bivariate Kumaraswamy model is not appropriate since it has support $0 < w_1 < w_2 < 1$ (if we were willing to accept the constraint $\delta_1 = \delta_2$). If the δ' s are unequal then the support set is unusual and it is difficult to envision a data set for which such a model will be appropriate.
- For the Dirichlet- bivariate Kumaraswamy model, the situation is similar.

To check the goodness of fit of all four statistical models, a χ^2 goodness-of-fit statistic is used and is computed using the computational package Mathematica. The MLEs are computed using the Nmaximize technique.

Table 1. Parameter estimates for HDI data set.

Model	Model I	Model II	Model III	Model IV
Parameter Estimates	$\hat{\alpha} = 3.5287(0.3335)$ $\hat{\delta_1} = 1.1845(0.9723)$ $\hat{\delta_2} = 3.2424(0.1065)$	$\hat{\alpha_1} = 0.692(0.0894)$ $\hat{\alpha_2} = 1.362(2.246)$ $\hat{p_1} = 3.016(0.9852)$ $\hat{p_2} = 0.782(5.681)$ $\hat{q} = 1.2218(0.3678)$	$\hat{\alpha} = 1.798(0.1142)$ $\hat{\beta} = 1.834(0.2794)$ $\hat{\gamma} = 4.038(0.3677)$ $\hat{\delta} = 0.587(1.2468)$	$\hat{\alpha_0} = 4.016(0.6436)$ $\hat{\alpha_1} = 3.7649(2.1873)$ $\hat{\alpha_2} = 6.172(0.5837)$
Log likelihood	-168.45	-205.38	-217.63	-196.39
χ^2 goodness p-value	0.6132	0.4821	0.4593	0.5041

For this particular data set, it appears that the best model, of the four that were considered, is the Libby- Novick-Jones-Olkin-Liu bivariate Kumaraswamy model.

10. AN ALTERNATIVE APPROACH USING COPULAS

The bivariate Kumaraswamy models discussed in this paper are constructed by focusing on bivariate beta random variables with the first parameter of each marginal beta distribution equal to one. An alternative approach, still using the Arnold-Ng bivariate model, is available.

Many researchers make use of what are called copula based bivariate models. For such models, one begins with a copula, a bivariate distribution with $Uniform(0,1)$ marginals, and makes marginal transformations to obtain a bivariate model with desired marginal distributions. The dependence structure of the resulting model is thus “inherited” from that of the particular copula used in the construction. Typically, one parameter families of copulas are used to build models in this way. More flexible models can be expected if multiparameter families of copulas are used.

A copula based bivariate Kumaraswamy model will be of the form

$$(X_1, X_2) = (\left[1 - (1 - Y_1)^{1/\delta_1}\right]^{1/\gamma_1}, \left[1 - (1 - Y_2)^{1/\delta_2}\right]^{1/\gamma_2}), \tag{10.1}$$

where (Y_1, Y_2) has the desired copula as its distribution (with $Uniform(01)$ marginals).

In (10.1) each X_i has ben obtained from the corresponding Y_i by transforming using a Kumaraswamy quantile function, to obtain Kumaraswamy marginals.

Looking baback at the Arnold-Ng bivariate beta model (3.1)-(3.2), it is evident that it contains many distributions with $Uniform(0,1)$ marginals since a $Uniform(0,1)$ can be identified as a $Beta(1,1)$ distribution. In fact the Arnold-Ng model contains a four parameter family of such distributions, i.e., of copulas. The subfamily of of the Arnold-Ng distributions that correspond to copulas is obtained by setting

$$\alpha_1 + \alpha_5 + \alpha_7 = 1, \tag{10.2}$$

$$\alpha_2 + \alpha_5 + \alpha_8 = 1, \tag{10.3}$$

$$\alpha_3 + \alpha_6 + \alpha_8 = 1, \tag{10.4}$$

$$\alpha_4 + \alpha_6 + \alpha_7 = 1. \tag{10.5}$$

In addition, recall that all α_i's are non-negative. The resulting four dimensional parameter space may be described as follows:

$$\alpha_5 \in [0,1], \alpha_6 \in [0,1], 0 \le \alpha_7 \le 1 - \max\{\alpha_5, \alpha_6\}, 0 \le \alpha_8 \le 1 - \max\{\alpha_5, \alpha_6\}.$$

The remaining α_i's, $i = 1,2,3,4$, are then determined by equations (10.2)-(10.5).

Such models will be referred to as Arnold-Ng (henceforth AN) copulas.

In a separate report, Arnold and Ghosh (2016) investigate the use of this multiparameter family of copulas in the construction of eight parameter bivariate Kumaraswamy models. The enhanced flexibility of a four parameter copula model, when compared with typical one parameter families, makes such an approach an attractive alternative. See Arnold and Ghosh (2016) for detailed discussion of all submodels (with one, two or three parameters) of the AN four parameter copula family. These can be used to construct (using (10.1)) five, six and seven parameter bivariate Kumaraswamy distributions.

11. CONCLUDING REMARKS

In this paper we consider several different strategies for constructing bivariate beta (and also bivariate generalized beta) distributions as well as several types of bivariate Kumaraswamy distributions using the gamma based methodology for construction of bivariate beta models as suggested by Arnold and Ng (2011). It has been observed that for most of the constructed bivariate beta models, a corresponding closed form expression for the joint density is unavailable. Our proposed bivariate beta models are significantly different than those discussed and studied in detail in Sarabia et al. (2014).

However, one can readily simulate data from those models using an appropriate algorithm. We have also constructed various bivariate Kumaraswamy models starting from a 8 parameter bivariate Kumaraswamy models by setting the first parameter for the associated beta random variables to 1 and then making a positive power transformation. During the discussion, we have considered some structural properties of the resultant models, such as moments, dependence structure, etc.

However, in many applications it might be desirable to first test the hypothesis $H : \delta_1 = \delta_2 = 1$, using perhaps a generalized likelihood ratio test, before settling on the use of a bivariate Kumaraswamy model as opposed to a bivariate beta or generalized beta model. A preliminary visual inspection of the sample marginals might be a useful first step. Bivariate beta and bivariate Kumaraswamy) distributions could play a useful role in modeling dependent risks (in a typical financial setting) where the individual risks are transformed to be bounded on the interval $[0, 1]$.

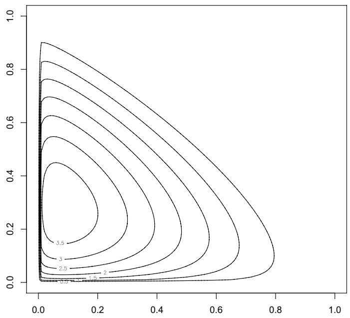
Estimation of the model parameters (especially, for those models without a closed form of the density) using an approximate Bayesian approach as well as using an appropriate method of moments strategy (using marginal, joint and/or conditional moments) is currently under investigation and, as noted in Section 8, will be reported elsewhere.

Bivariate Kumaraswamy distributions might be considered as models in certain bivariate reliability contexts. Howver, the absence of corresponding density functions will typically not allow one to identify bivariate failure rate functions and other distributional features of interest in reliability. Numerical evaluations or simulation based approximations will be needed in almost all cases. One case in which a density exists is the Libby- Novick-Jones-Olkin-Liu bivariate Kumaraswamy distribution, displayed in (6.5). In this case, for example, it is possible to obtain a rather complicated series expansion for the reliability quantity $P(W_1 < W_2)$. See Appendix B. Expressions for other reliability features can be expected to be equally or more complicated and, even as in this simple case, will be of doubtful utility.

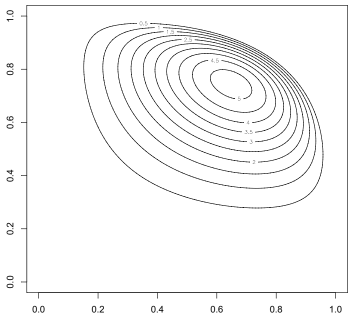
APPENDIX A

In Figure 1 we provide contour plots for some specific choices of the parameters α and δ_j for $j = 1, 2$ for some representative 3 parameter BK models. The following choices are made for each of these selected representative 3 parameter BK models:

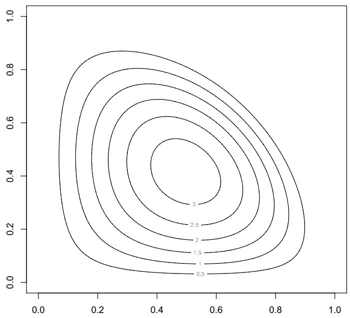
- Choice 1 (c1): $\alpha = 1.2, \delta_1 = 0.5, \delta_2 = 0.5$.
- Choice 2 (c2): $\alpha = 1.8, \delta_1 = 1.3, \delta_2 = 0.9$.



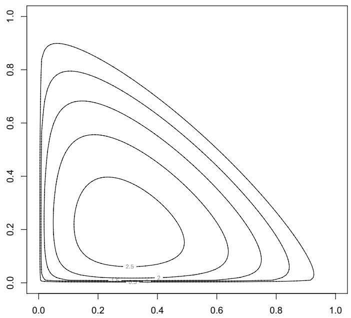
Dirichlet BK contours(c1)



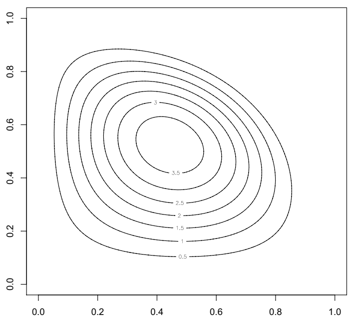
Dirichlet BK contours(c2)



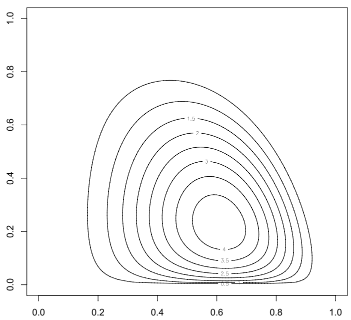
NK-BK contours(c1)



NK-BK contours(c2)



LNJOL-BK contours(c1)



LNJOL-BK contours(c2)

Figure 1: Contour plots for representative BK models.

APPENDIX B

The joint density of the LNJOLBK distribution is of the form

$$f_{\underline{W}}(\underline{w}) = \alpha(\alpha+1)\delta_1\delta_2 w_1^{\delta_1-1} w_2^{\delta_2-1} \frac{(1-w_1^{\delta_1})^{\alpha}(1-w_2^{\delta_2})^{\alpha}}{(1-w_1^{\delta_1}w_2^{\delta_2})^{\alpha+2}} \; I(0<w_1,w_2<1).$$

In this case,

$$R = P(W_1 < W_2) = \int_0^1 \int_{w_1}^1 f(w_1, w_2)\, dw_2 dw_1.$$

First, let us consider the integral

$$
\begin{aligned}
I_1 &= \int_{w_1}^{1} \delta_2 w_2^{\delta_2-1} \frac{(1-w_2^{\delta_2})^{\alpha}}{(1-w_1^{\delta_1} w_2^{\delta_2})^{\alpha+2}} dw_2 \\
&= \int_{w_1^{\delta_2}}^{1} (1-t)^{\alpha} \left(1-t w_1^{\delta_1}\right)^{-(\alpha+2)} dt, \quad \text{on substitution} t = w_1^{\delta_1} \\
&= \sum_{k=0}^{\infty} w_1^{k\delta_1} \binom{\alpha+2+k-1}{k} \int_{w_1^{\delta_2}}^{1} t^k (1-t)^{\alpha} dt
\end{aligned}
$$

using the expansion

$$
(1-z)^{-m} = \sum_{k=0}^{\infty} \binom{m+k-1}{k} z^k.
$$

Next, consider the integral on I_1

$$
\int_{w_1^{\delta_2}}^{1} t^k (1-t)^{\alpha} dt = B(k+1, \alpha-1) - \left(w_1^{\delta_1}\right)^{k+1} \sum_{n=0}^{\infty} \frac{(1-\alpha)_{(n)} w_1^{n\delta_1}}{n!(k+n)},
$$

using the series expansion of the incomplete Beta function

$$
B(z,a,b) = \int_0^z u^{a-1}(1-u)^{b-1} du = z^a \sum_{n=0}^{\infty} \frac{(1-b)_{(n)} z^n}{n!(a+n)},
$$

where $T_{(n)}$ is the descending factorial.

Hence, the expression I_1 reduces to

$$
I_1 = \sum_{k=0}^{\infty} \binom{\alpha+2+k-1}{k} B(k+1, \alpha-1) w_1^{k\delta_1+\delta_2} - \sum_{k=0}^{\infty} \sum_{n=0}^{\infty} \frac{w_1^{(2k+n+1)\delta_1} (1-\alpha)_{(n)}}{n!(k+n)}.
$$

Therefore, the expression of R, the reliability parameter for this bivariate KW model can be expressed in the form

$$
\begin{aligned}
R &= \int_0^1 \alpha(\alpha+1)\delta_1 w_1^{\delta_1-1} \left(1-w_1^{\delta_1}\right)^{\alpha} I_1 dw_1 \\
&= \alpha(\alpha+1) \left[\sum_{k=0}^{\infty} \binom{\alpha+2+k-1}{k} B(k+1, \alpha-1) \delta_1 \int_0^1 w_1^{\delta_1(1+k)+\delta_2-1} \right. \\
&\qquad \times \left(1-w_1^{\delta_1}\right)^{\alpha} dw_1 \\
&\quad \left. - \sum_{k=0}^{\infty} \sum_{n=0}^{\infty} \frac{(1-\alpha)_{(n)}}{n!(k+n)} \delta_1 \int_0^1 w_1^{\delta_1(2+2k+n)-1} \left(1-w_1^{\delta_1}\right)^{\alpha} dw_1 \right] \\
&= \alpha(\alpha+1) \left[\sum_{k=0}^{\infty} \binom{\alpha+2+k-1}{k} B(k+1, \alpha-1) B(k\delta_1+\delta_2+1, \alpha+1) \right. \\
&\quad \left. - \sum_{k=0}^{\infty} \sum_{n=0}^{\infty} \frac{a}{n!(k+n)} B(2k+n+2, \alpha+1) \right],
\end{aligned}
$$

provided $\alpha > 1$.

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